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INFLUENCE OF THE HIGHER ORDER DERIVATIVES ON THE PLANET PERIHELION PRECESSION IN THE EINSTEIN FIELD EQUATIONS FOR VACUUM CONDITION

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Abstract

This paper studies the effect of higher order derivative tensor in the Einstein field equations for vacuum condition on the planet perihelion precession. This tensor was initially proposed as the space-time curvature tensor by Deser and Tekin on discussions about the energy effects caused by this tensor. However, they include this tensor to Einstein field equations as a new model in general relativity theory. This is very interesting since there are some questions in cosmology and astrophysics that have no answers. Thus, they hoped this model could solve those problems by finding analytical or perturbative solution and interpreting it. In this case, the perturbative solution was used to find the Schwarzschild solution and it was also applied to consider the planetary motion in the solar gravitational field. Furthermore, it was proven that the tensor is divergence-free in order to keep the Einstein field equations remain valid.

Keywords: astrophysical model, higher order derivative tensor, perihelion precession, perturbative solution of Schwarzschild metric

1. Introduction

Until today, there remain some problems in astrophysics with unsatisfactory answers. One of them is the issue of determining a general theory to connect theory calculations with observation results of planet perihelion precession, for those located far from the sun. The general theory of relativity is considered, until now, as the strongest candidate to explain the natural phenomenon; however, the calculations of the perihelion precession work for observation results for planets located near the sun only, e.g the planet Mercury. There have been a lot of assumptions suggested by physicists to explain the condition. One of the assumptions suggests that because Mercury is the closest planet to the sun, 'one-body problem' approach can be used to study the movement of the planet. One of the ways to solve the problem is by modifying Einstein field equations. The first person to modify it was de Sitter, by adding a cosmological constant.

The cosmological constant was first suggested by Einstein, who initially intended to connect general theory of relativity and Mach's principle of inertia. At the time, Einstein rejected it as it contradicted Mach's principle which requires no solution for vacuum condition. However, de Sitter found that by adding a cosmological constant in Einstein field equations for vacuum condition, it shows the possibility of curved space time, which means that Einstein equations has a solution [1].

This later opened the possibilities of modifying Einstein field equations to solve the aforementioned problems in astrophysics. In 2003, Deser and Tekin suggested higher order derivatives which were first intended as materials for new studies in cosmology and astrophysics [2-5] (other models have been proposed in other ways too, see also [6-8]. This additional term initially introduced by Deser and Tekin as a curvature tensor and applying it to discuss the definition of its energy-momentum tensor [9]. This paper studies the Schwarzschild solution for Einstein field equations in vacuum condition with higher order derivatives tensor using Frobenius method, and later applies it to analyze planet perihelion precession. It is interesting as perihelion precession of a planet orbiting the sun is almost identical with that of planets located very close to the sun; therefore, by having the term as an addition opens up the possibilities of inferring precession corrections for planets that orbit far from the sun.

2. Methods

Convention used for metric tensor in this paper is $(+---)$. Einstein field equations, non-de Sitter, with higher order derivatives in vacuum condition form: $(\mu,\kappa = 0,1,2,3)$ [2-5].

$$R_{\mu\kappa} - \frac{1}{2} R g_{\mu\kappa} + \Phi_{\mu\kappa} = 0, \tag{1}$$

with definition of tensor

$$\begin{aligned} \Phi_{\mu\kappa} &= 2\alpha R\left(R_{\mu\kappa} - \frac{1}{4} R g_{\mu\kappa} \right) \\ &+ (2\alpha+\beta)\left(g_{\mu\kappa} \Box - \nabla_\mu \nabla_\kappa \right) R \\ &+ \beta \Box \left(R_{\mu\kappa} - \frac{1}{2} R g_{\mu\kappa} \right) \\ &+ 2\beta\left(R_{\mu\sigma\kappa\rho} - \frac{1}{4} g_{\mu\kappa} R_{\sigma\rho} \right) R^{\sigma\rho}, \end{aligned} \tag{2}$$

and d'Alembertian operator is defined as $\Box = \nabla^\tau \nabla_\tau$. In equation (2), α dan β constants are arbitrary, thus for $\alpha = \beta = 0$, the equation above becomes Einstein field equations in vacuum condition. Moreover, it can be shown that $\Phi_{\mu\kappa}$ is divergence free.

$$\nabla^\kappa \Phi_{\mu\kappa} = 0. \tag{3}$$

Firstly, we consider Schwarzschild metric in static field and spherical symmetry in relativistic units of

$$\begin{aligned} ds^2 &= e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 \\ &- r^2 \sin^2 \theta d\phi^2. \end{aligned} \tag{4}$$

To find perturbative solution of the equation above, we initially look for the vacuum solution without higher order derivatives. Vacuum condition without higher order derivatives satisfies the equation of

$$R_{\mu\kappa} = 0, \tag{5}$$

with the following non-zero Ricci tensor components

$$R_{00} = e^{\nu-\lambda}\left(\frac{\nu'^2}{4} - \frac{1}{4}\lambda'\nu' + \frac{\nu''}{2} + \frac{\nu'}{r} \right), \tag{6}$$

$$R_{11} = -\frac{\nu''}{2} + \frac{1}{4}\lambda'\nu' - \frac{\nu'^2}{4} + \frac{\lambda'}{r}, \tag{7}$$

$$R_{22} = \left\{ \left(-\frac{1}{2}\nu' r + \frac{1}{2}\lambda' r - 1 \right) e^{-\lambda} + 1 \right\}, \tag{8}$$

$$R_{33} = R_{22} \sin^2 \theta. \tag{9}$$

Moreover, the obtained Ricci scalar:

$$\begin{aligned} R &= e^{-\lambda}\left(\frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu' + \nu'' + 2\frac{\nu'}{r} - 2\frac{\lambda'}{r} + \frac{2}{r^2} \right) \\ &- \frac{2}{r^2}. \end{aligned} \tag{10}$$

The sign ($'$) in the five equations suggest derivatives of r . If the condition of equation (5) is substituted to equation (2), the following connection is satisfied

$$\Phi_{\mu\kappa}\Big|_{R_{\mu\kappa}=0} = 0. \tag{11}$$

The non-zero terms in the equation above are [10]

$$\begin{aligned} \Phi_{00} = \alpha\Bigg\{ & e^{\nu-2\lambda}\Bigg(\frac{1}{8}\nu'^4 - \frac{1}{4}\lambda'\nu'^3 + \frac{1}{2}\nu'^2\nu'' + \frac{\nu'^3}{r} \\ & - \frac{11}{8}\nu'^2\lambda'^2 + \frac{9}{2}\nu'\nu''\lambda' - \frac{3}{2}\nu''^2 + \frac{\nu'^2\lambda'}{r} \\ & - 2\frac{\nu'\nu''}{r} + 2\frac{\nu'^2}{r^2} - 6\frac{\lambda'^2}{r^2} - 8\frac{\lambda'}{r^3} - \frac{10}{r^4} \\ & + \frac{3}{2}\lambda'^3\nu' - \frac{11}{2}\nu''\lambda'^2 - 8\frac{\nu'\lambda'^2}{r} + 6\frac{\lambda'^3}{r} \\ & - \frac{7}{2}\lambda''\lambda'\nu' + 6\nu'''\lambda' + 16\frac{\lambda'\nu''}{r} - 2\frac{\nu'\lambda'}{r^2} \\ & - 14\frac{\lambda''\lambda'}{r} + 6\frac{\lambda''\nu'}{r} - 8\frac{\nu'''}{r} + 4\frac{\lambda''}{r^2} \\ & + \nu'^2\lambda'' + 4\nu''\lambda'' - 2\nu'\nu''' + \lambda'''\nu' \\ & - 2\nu'''' + 4\frac{\lambda'''}{r} \Bigg) + 12\frac{e^{\nu-\lambda}}{r^4} - 2\frac{e^{\nu}}{r^4} \Bigg\} \\ + \beta\Bigg\{ & e^{\nu-2\lambda}\Bigg(-\frac{11}{16}\nu'^2\lambda'^2 + \frac{3}{4}\lambda'^3\nu' - \frac{11}{4}\lambda'^2\nu'' \\ & - \frac{13}{4}\frac{\nu'\lambda'^2}{r} + \frac{3}{2}\frac{\lambda'^3}{r} + \frac{9}{4}\nu'\nu''\lambda' - \frac{7}{4}\nu'\lambda'\lambda'' \\ & + 3\nu'''\lambda' + \frac{15}{2}\frac{\lambda'\nu''}{r} - \frac{3}{2}\frac{\lambda'\nu'}{r^2} - \frac{7}{2}\frac{\lambda''\lambda'}{r} \\ & - 2\frac{\lambda'}{r^3} + \frac{3}{2}\frac{\lambda'\nu'^2}{r} - \frac{7}{4}\frac{\lambda'^2}{r^2} - \frac{5}{2}\frac{\nu'\nu''}{r} \\ & + \frac{5}{2}\frac{\lambda''\nu'}{r} - 4\frac{\nu'''}{r} + \frac{\lambda''}{r^2} - \frac{3}{r^4} + \frac{1}{2}\nu'^2\lambda'' \\ & + 2\lambda''\nu'' - \nu'\nu''' - \frac{3}{4}\nu''^2 + \frac{1}{2}\lambda'''\nu' \\ & - \nu'''' + \frac{\lambda'''}{r} + \frac{1}{4}\frac{\nu'^3}{r} + \frac{1}{16}\nu'^4 \\ & - \frac{1}{8}\nu'^3\lambda' + \frac{1}{4}\nu'^2\nu'' + \frac{1}{4}\frac{\nu'^2}{r^2} \Bigg) \\ & + 4\frac{e^{\nu-\lambda}}{r^4} - \frac{e^{\nu}}{r^4} \Bigg\}, \end{aligned} \tag{12}$$

$$\begin{aligned} \Phi_{11} = \alpha\Bigg\{ & e^{-\lambda}\Bigg(\frac{1}{2}\nu'^2\nu'' - \frac{1}{4}\lambda'\nu'^3 - \frac{1}{8}\nu'^4 - \nu'\nu''\lambda' \\ & + \frac{3}{8}\nu'^2\lambda'^2 + 3\frac{\nu'\lambda'^2}{r} - \frac{1}{2}\nu''^2 - 4\frac{\nu''\lambda'}{r} \\ & - 3\frac{\nu'^2\lambda'}{r} - 8\frac{\nu'}{r^3} + 6\frac{\lambda'^2}{r^2} - \frac{14}{r^4} - \frac{1}{2}\lambda''\nu'^2 \\ & + \nu'\nu''' + 6\frac{\nu'\nu''}{r} - 4\frac{\nu'\lambda''}{r} - 8\frac{\lambda'\nu'}{r^2} \\ & + 4\frac{\nu'''}{r} + 8\frac{\nu''}{r^2} - 8\frac{\lambda''}{r^2} \Bigg) + 2\frac{e^{\lambda}}{r^4} + \frac{12}{r^4} \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&+\beta\left\{e^{-\lambda}\left(-\frac{1}{8}\nu'^3\lambda'+\frac{3}{16}\lambda'^2\nu'^2-\frac{1}{2}\lambda'\nu'\nu''\right.\right.\\
&-\frac{\nu'^2\lambda'}{r}+\frac{3}{4}\frac{\lambda'^2\nu'}{r}+\frac{1}{4}\nu'^2\nu''-\frac{1}{4}\nu'^2\lambda''\\
&+\frac{1}{2}\nu'''\nu'+2\frac{\nu'\nu''}{r}-\frac{3}{4}\frac{\nu'^2}{r^2}-\frac{\lambda''\nu'}{r}\\
&-2\frac{\nu'}{r^3}-\frac{\lambda'\nu''}{r}-\frac{5}{2}\frac{\nu'\lambda'}{r^2}+\frac{9}{4}\frac{\lambda'^2}{r^2}+\frac{\nu'''}{r}\\
&+2\frac{\nu''}{r^2}-3\frac{\lambda''}{r^2}-\frac{5}{r^4}+\frac{1}{4}\frac{\nu'^3}{r}-\frac{1}{16}\nu'^4\\
&\left.\left.-\frac{1}{4}\nu''^2\right)+\frac{e^{\lambda}}{r^4}+\frac{4}{r^4}\right\},
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\Phi_{22}=\alpha&\left\{e^{-2\lambda}\left(\frac{1}{8}\nu'^4r^2-\frac{3}{4}\lambda'\nu'^3r^2-4\lambda'''r\right.\right.\\
&+\frac{17}{8}\nu'^2\lambda'^2r^2-7\nu'\nu''\lambda'r^2+\frac{1}{2}\nu'^3r\\
&-4\lambda'\nu'^2r+5\nu'\nu''r-2\frac{\nu'}{r}+\frac{19}{2}r\lambda'^2\nu'\\
&+\frac{5}{2}\nu''^2r^2-14\nu''\lambda'r+14\frac{\lambda'}{r}+\frac{14}{r^2}\\
&-\frac{3}{2}\lambda''\nu'^2r^2+3\nu'\nu'''r^2-2\nu'^2\\
&-7\lambda''\nu'r-\frac{3}{2}\lambda'^3\nu'r^2+\frac{11}{2}\lambda'^2\nu''r^2\\
&-6\lambda'^3r+\frac{7}{2}\lambda''\nu'\lambda'r^2-6\lambda'\nu'''r^2\\
&+6\lambda'\nu'+14\lambda'\lambda''r+6\nu'''r-4\nu''\\
&-4\nu''\lambda''r^2-\lambda'''\nu'r^2-2\nu''''r^2\\
&\left.+\frac{3}{2}\nu'^2\nu''r^2\right)\\
&\left.+e^{-\lambda}\left(6\frac{\nu'}{r}-6\frac{\lambda'}{r}-\frac{12}{r^2}\right)-\frac{2}{r^2}\right\}\\
+\beta&\left\{e^{-2\lambda}\left(-\frac{1}{4}\nu'^3\lambda'r^2+\frac{9}{16}\lambda'^2\nu'^2r^2\right.\right.\\
&-\frac{15}{8}\lambda'\nu'\nu''r^2-\frac{3}{4}\nu'^2\lambda'r+3r\lambda'^2\nu'\\
&+\frac{1}{2}\nu'^2\nu''r^2-\frac{3}{8}\nu'^2\lambda''r^2+\frac{3}{4}\nu'\nu'''r^2\\
&+\frac{3}{4}\nu''\nu'r-\frac{1}{2}\nu'^2-\frac{9}{4}\nu'\lambda''r-\frac{3}{8}\lambda'^3\nu'r^2\\
&+\frac{11}{8}\nu''\lambda'^2r^2-\frac{9}{4}\lambda'^3r+\frac{7}{8}\lambda'\lambda''\nu'r^2\\
&-\frac{3}{2}\nu'''\lambda'r^2-\frac{15}{4}\nu''\lambda'r+\frac{3}{2}\lambda'\nu'\\
&+\frac{21}{4}\lambda'\lambda''r+5\frac{\lambda'}{r}+\frac{3}{2}\nu'''r-\nu''\\
&+\frac{5}{r^2}-r^2\nu''\lambda''+\frac{3}{4}\nu''^2r^2-\frac{1}{4}\lambda'''\nu'r^2\\
&\left.+\frac{1}{2}\nu''''r^2-\frac{3}{2}\lambda'''r-\frac{\nu'}{r}+\frac{1}{16}\nu'^4r^2\right)\\
&\left.+e^{-\lambda}\left(2\frac{\nu'}{r}-2\frac{\lambda'}{r}-\frac{4}{r^2}\right)-\frac{1}{r^2}\right\},
\end{aligned}
\tag{14}
$$

$$\Phi_{33}=\Phi_{22}\sin^2\theta. \tag{15}$$

Moreover, by applying condition (5) the following solution is obtained

$$\nu_0(r)=\ln\left(1-\frac{2m}{r}\right), \tag{16}$$

$$\lambda_0(r)=-\ln\left(1-\frac{2m}{r}\right), \tag{17}$$

with $\nu_0(r)$ and $\lambda_0(r)$ as "background" solution for vacuum condition without higher order derivatives. Perturbative solution is obtained by stating that the general solutions to $\nu(r)$ and $\lambda(r)$ are in forms of

$$\nu(r)=\nu_0(r)+\nu_1(r), \tag{18}$$

$$\lambda(r)=\lambda_0(r)+\lambda_1(r), \tag{19}$$

suggesting $\nu_1(r)$ and $\lambda_1(r)$ as minor terms. The method of problem solving above is actually taken by considering the analogy in ref.[11]. This paper studies only the case of $\alpha\neq 0$ dan $\beta=0$.

Next, we expand the terms $\nu_1(r)$ and $\lambda_1(r)$ into power series of [12, 13]

$$\nu_1(r)=\sum_{n=0}^{\infty}a_n r^{n+s}, \tag{20}$$

$$\lambda_1(r)=\sum_{n=0}^{\infty}b_n r^{n+s}, \tag{21}$$

with *s* a determined constant, while a_n and b_n are the coefficients sought. By adding all non-zero tensor components into equation (1), continued by including equations (18), (19), (20), and (21) the obtained value *s* = 0, and it can also be shown that the non-trivial solution is obtained by getting $a_0\neq 0$, $a_n=0$ (for *n* > 0), and $b_n=0$ (for all *n*). Therefore, we get the following relations

$$\nu(r)=\nu_0(r)+a_0, \tag{22}$$

$$\lambda(r)=\lambda_0(r). \tag{23}$$

Schwarzschild solution can be obtained by substituting equation (22) and (23) to equation (4)

$$
\begin{aligned}
ds^2=C\left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2\\
-r^2d\theta^2-r^2\sin^2\theta d\phi^2,
\end{aligned}
\tag{24}
$$

with *C* constant.

3. Results and Discussion

This step discusses planet perihelion precession with Schwarzschild solution which satisfies equation (24). For further discussion, we survey "timelike" geodesic obtained by differentiating (24) with "proper time"

$$K = C\left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 = 1. \quad (25)$$

The sign (.) in equation (25) suggests derivatives of "proper time". Geodesic equation of planet movement is given using the following Euler-Lagrange equation

$$\frac{\partial K}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial K}{\partial \dot{x}^\mu}\right) = 0, \quad (26)$$

with τ "proper time". Next, by using $\mu = 0,2,3$ in equation (26) three following equations of motions are obtained

$$\frac{d}{d\tau}\left[(1-2m/r)\dot{t}\right] = 0, \quad (27)$$

$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right) - r^2\sin\theta\cos\theta\dot{\phi}^2 = 0, \quad (28)$$

$$\frac{d}{d\tau}\left(r^2\sin^2\theta\dot{\phi}\right) = 0. \quad (29)$$

The three equations above are without *C* constant terms printed on equation (24), so the influence of higher order tensor for the above solution does not work on planet perihelion precession. For further discussion, by having limitation of motion of $\theta = \pi/2$, thus after the planet has fully evolved ($\phi = 2\pi$) perihelion precession for the planet can be obtained from the above three equations by using perturbative method in relativistic unit of

$$\Delta\phi = \frac{6\pi m}{a(1-e^2)}, \quad (30)$$

with *a* and *e* of each major and eccentricity ellipse axis length. The solution of perihelion precession itself is considered more detail in many textbooks on general relativity, for example in ref. [14-17]. Complete derivatives of equation (30) are attached.

4. Conclusion

Einstein field of equations can be modified by adding any tensor term, requiring free-divergence for symmetrical metric tensor. This paper, mathematically, adds tensor term with free-divergence higher order derivatives in Einstein field equations initially introduced by Deser and Tekin as a curvature tensor. This tensor depends on two independent constants which can be chosen by considering some aspects.

This paper also shows that the addition of higher order derivative tensor in Einstein field equations for vacuum condition does not result in planet perihelion precession through perturbative solution using Frobenius method. However, the addition of the mentioned tensor terms can start new studies in cosmology and astrophysics.

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Appendix

To calculate planet perihelion precession, we apply boundary condition of $\theta = \pi/2$ and $\dot{\theta} = 0$. Thus, equation (28) is satisfied in trivial with equation (27) and (29), each reduced into

$$\left(1-\frac{2m}{r}\right)\dot{t} = \gamma, \quad (31)$$

$$r^2\dot{\phi} = h, \quad (32)$$

with γ and *h* as integrated constant. Next, we transform variable $r = 1/u$ and substitute $\dot{t}$ and $\dot{\phi}$ in equation (31) and (32) to equation (25) by stating in variable u and ϕ, which result in

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{C\gamma^2 - 1}{h^2} + \frac{2m}{h^2}u + 2mu^3. \quad (33)$$

To facilitate calculation, we differentiate equation (33) to ϕ

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2. \quad (34)$$

The perturbative solution is resulted by seeing first comparison of $3mu^2$ with m/h^2, which has minor value of $3h^2u^2$ [14]. For this reason, we introduce small parameter of $\varepsilon = 3m^2/h^2$ and assume general solution in the form of

$$u = u_0 + \varepsilon u_1. \quad (35)$$

After substituting equation (35) to equation (34), we then obtain equation for correction term of

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{h^2}{m}u_0{}^2, \quad (36)$$

with solution $u_0 = \frac{m}{h^2}(1 + e\cos\phi)$. Next, we obtain trial solution to the left-hand side of equation (36) in the form of

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi. \quad (37)$$

Therefore, general solution is resulted in the form of

$$u = \frac{m}{h^2}(1 + e\cos\phi) + \varepsilon\left(\frac{m}{h^2}\left(1 + e^2/2\right)\right.$$
$$\left.\frac{me}{h^2}\phi\sin\phi - \frac{me^2}{6h^2}\cos^2\phi\right). \quad (38)$$

Continuous planet revolution causes the term $\phi\sin\phi$ dominant, and by using trigonometry algebra and Taylor expansion for small parameter, they result in this solution

$$u \approx \frac{m}{h^2}\left(1 + e\cos(\phi - \varepsilon\phi)\right). \quad (39)$$

Planet perihelion precession after evolution is $\phi = 2\pi$ defined by

$$\Delta\phi = \varepsilon\phi = \frac{6\pi m^2}{h^2}, \quad (40)$$

with $h^2 = a(1 - e^2)m$.

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