LAMPIRAN

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LAMPIRAN 1

Menunjukkan: θ memaksimumkan $L(\theta) \leftrightarrow \theta$ memaksimumkan $\ln L(\theta)$

Bukti:

($\rightarrow$) karena θ memaksimumkan $L(\theta)$ maka

- $\dfrac{\partial L(\theta)}{\partial \theta} = 0$

$$\frac{\partial L(\theta)}{\partial \theta} = 0 \quad \leftrightarrow \quad \frac{1}{L(\theta)} . \frac{\partial L(\theta)}{\partial \theta} = \frac{1}{L} .0$$

$$\leftrightarrow \quad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

- $\dfrac{\partial^2 L(\theta)}{\partial \theta^2} < 0$

$$\frac{\partial^2 L(\theta)}{\partial \theta^2} < 0 \quad \leftrightarrow \quad \frac{\partial^2 L(\theta)}{\partial \theta^2} . \frac{1}{L(\theta)} < 0 . \frac{1}{L(\theta)}$$

$$\leftrightarrow \quad \frac{\partial^2 L(\theta)}{\partial \theta^2} . \frac{1}{L(\theta)} < 0$$

$$\leftrightarrow \quad \frac{\partial}{\partial \theta}\left(\frac{1}{L(\theta)}\right) . \frac{\partial L(\theta)}{\partial \theta} + \frac{\partial^2 L(\theta)}{\partial \theta^2} . \frac{1}{L(\theta)} < \frac{\partial}{\partial \theta}\left(\frac{1}{L(\theta)}\right) . \frac{\partial L(\theta)}{\partial \theta} + 0$$

$$\leftrightarrow \quad \frac{\partial}{\partial \theta}\left(\frac{1}{L(\theta)}\right) . \frac{\partial L(\theta)}{\partial \theta} + \frac{\partial^2 L(\theta)}{\partial \theta^2} . \frac{1}{L(\theta)} < \frac{\partial}{\partial \theta}\left(\frac{1}{L(\theta)}\right) .0 + 0$$

$$\leftrightarrow \quad \frac{\partial}{\partial \theta}\left(\frac{1}{L(\theta)}\right) . \frac{\partial L(\theta)}{\partial \theta} + \frac{\partial^2 L(\theta)}{\partial \theta^2} . \frac{1}{L(\theta)} < 0$$

$$\leftrightarrow \quad \frac{\partial}{\partial\theta}\left(\frac{1}{L(\theta)}.\frac{\partial L(\theta)}{\partial\theta}\right) < 0$$

$$\leftrightarrow \quad \frac{\partial}{\partial\theta}\left(\frac{\partial \ln L(\theta)}{\partial\theta}\right) < 0$$

$$\leftrightarrow \quad \frac{\partial^2 \ln L(\theta)}{\partial\theta^2} < 0$$

(→) Terbukti bahwa jika θ memaksimumkan $L(\theta)$ maka θ memaksimumkan $\ln L(\theta)$.

(←) karena θ memaksimumkan $\ln L(\theta)$ maka

- $\frac{\partial \ln L(\theta)}{\partial\theta} = 0$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = 0 \quad \leftrightarrow \quad \frac{1}{L(\theta)}.\frac{\partial L(\theta)}{\partial\theta} = 0$$

$$\leftrightarrow \quad L.\frac{1}{L(\theta)}.\frac{\partial L(\theta)}{\partial\theta} = L.0$$

$$\leftrightarrow \quad \frac{\partial L(\theta)}{\partial\theta} = 0$$

- $\frac{\partial^2 \ln L(\theta)}{\partial\theta^2} < 0$

$$\frac{\partial^2 \ln L(\theta)}{\partial\theta^2} < 0 \quad \leftrightarrow \quad \frac{\partial}{\partial\theta}\left(\frac{1}{L(\theta)}.\frac{\partial L(\theta)}{\partial\theta}\right) < 0$$

$$\leftrightarrow \quad \frac{\partial}{\partial\theta}\left(\frac{1}{L(\theta)}\right).\frac{\partial L(\theta)}{\partial\theta} + \frac{\partial^2 L(\theta)}{\partial\theta^2}.\frac{1}{L(\theta)} < 0$$

$$\leftrightarrow \quad \frac{\partial}{\partial\theta}\left(\frac{1}{L(\theta)}\right).0 + \frac{\partial^2 L(\theta)}{\partial\theta^2}.\frac{1}{L(\theta)} < 0$$

$$\leftrightarrow \quad \frac{\partial^2 L(\theta)}{\partial \theta^2} \cdot \frac{1}{L(\theta)} < 0$$

$$\leftrightarrow \quad \frac{\partial^2 L(\theta)}{\partial \theta^2} \cdot \frac{1}{L(\theta)} . L(\theta) < 0 . L(\theta)$$

$$\leftrightarrow \quad \frac{\partial^2 L(\theta)}{\partial \theta^2} < 0$$

(←) Terbukti bahwa jika θ memaksimumkan $\ln L(\theta)$ maka θ memaksimumkan $L(\theta)$.

$\therefore$ Terbukti bahwa θ memaksimumkan $L(\theta) \leftrightarrow \theta$ memaksimumkan $\ln L(\theta)$

LAMPIRAN 2

Menunjukkan : $-\log(x)$ adalah fungsi konveks

Suatu fungsi dikatakan fungsi konveks jika:

$\forall x_1, x_2 \in \text{Domain}(f) \rightarrow$ untuk $\lambda \in [0,1]$

berlaku $\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2)$

Misalkan $f(x) = -\log(x)$,

Maka $f'(x) = -\frac{1}{x}$

$f''(x) = \frac{1}{x^2}$

Karena x^2 selalu bernilai positif, maka $f''(x) = \frac{1}{x^2} > 0$

Berdasarkan teorema: “Jika $f(x)$ dapat diturunkan 2 kali dan $f''(x) \geq 0$ maka $f(x)$ adalah fungsi konveks”,

$\therefore$ Dapat dinyatakan bahwa $-\log(x)$ adalah fungsi konveks.

LAMPIRAN 3

Membuktikan pertidaksamaan Jensen :

" Jika *f* adalah fungsi konveks, dan untuk $\lambda_i \in [0,1]$ sedemikian

sehingga $\sum_{i=1}^{n} \lambda_i = 1$, maka berlaku $\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$ "

Bukti:

Misalkan *f* fungsi konveks, maka berdasarkan definisi fungsi konveks,

$\forall\ x_1, x_2 \in \text{Domain}(f)$, maka untuk $\lambda_i \in [0,1]$ berlaku

$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$.

Pembuktian pertidaksamaan Jensen akan dilakukan dengan induksi matematika.

- Akan dibuktikan untuk n=2 pertidaksamaan Jensen benar.

  Adib: Jika *f* fungsi konveks dan untuk $\lambda_i \in [0,1]$ sedemikian sehingga

  $$\sum_{i=1}^{2} \lambda_i = 1, \text{ maka } \sum_{i=1}^{2} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{2} \lambda_i x_i\right)$$

  Bukti:

  $$\begin{aligned}\sum_{i=1}^{2} \lambda_i f(x_i) &= \lambda_1 f(x_1) + \lambda_2 f(x_2) \\ &= \lambda_1 f(x_1) + (1-\lambda_1) f(x_2) && \leftarrow \text{karena} \sum_{i=1}^{n} \lambda_i = 1 \\ &\geq f(\lambda_1 x_1 + (1-\lambda_1)x_2) && \leftarrow \text{karena } f \text{ fungsikonveks} \\ &\geq f\left(\sum_{i=1}^{2} \lambda_i x_i\right)\end{aligned}$$

  Jadi, terbukti bahwa untuk n=2 pertidaksamaan Jensen benar.

- Misalkan pertidaksamaan Jensen benar untuk n.

  Berlaku:

  Jika *f* adalah fungsi konveks, dan untuk $\lambda_i \in [0,1]$ sedemikian

  sehingga $\sum_{i=1}^{n} \lambda_i = 1$, maka berlaku

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_1 + \cdots + \lambda_n x_n)$$

- Akan dibuktikan untuk n+1 pertidaksamaan Jensen juga benar.

  Adib: Jika *f* fungsi konveks dan untuk $\lambda_i \in [0,1]$ sedemikian sehingga

  $\sum_{i=1}^{n+1} \lambda_i = 1$, maka:

$$\sum_{i=1}^{n+1} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right)$$

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) + \lambda_{n+1} f(x_{n+1}) \geq f(\lambda_1 x_1) + f(\lambda_2 x_2) + \cdots + f(\lambda_n x_n) + f(\lambda_{n+1} x_{n+1})$$

  Bukti:

  Karena

$$\sum_{i=1}^{n+1} \lambda_i = 1 \quad \leftrightarrow \quad \sum_{i=1}^{n} \lambda_i = 1 - \lambda_{n+1} \leftrightarrow \quad \frac{\sum_{i=1}^{n} \lambda_i}{1-\lambda_{n+1}} = 1$$

$$\leftrightarrow \quad \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{1-\lambda_{n+1}} = 1$$

$$\leftrightarrow \quad \frac{\lambda_1}{1-\lambda_{n+1}} + \frac{\lambda_2}{1-\lambda_{n+1}} + \cdots + \frac{\lambda_n}{1-\lambda_{n+1}} = 1$$

Maka,

$$\frac{\lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_n f(x_n)}{1-\lambda_{n+1}} = \frac{\lambda_1}{1-\lambda_{n+1}}f(x_1)+\frac{\lambda_2}{1-\lambda_{n+1}}f(x_2)+\cdots+\frac{\lambda_n}{1-\lambda_{n+1}}f(x_n)$$
$$\geq f\left(\frac{\lambda_1}{1-\lambda_{n+1}}x_1+\frac{\lambda_2}{1-\lambda_{n+1}}x_2+\cdots+\frac{\lambda_n}{1-\lambda_{n+1}}x_n\right)$$

Jadi

$$\frac{\lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_n f(x_n)}{1-\lambda_{n+1}} \geq f\left(\frac{\lambda_1}{1-\lambda_{n+1}}x_1+\frac{\lambda_2}{1-\lambda_{n+1}}x_2+\cdots+\frac{\lambda_n}{1-\lambda_{n+1}}x_n\right)$$

Kalikan kedua ruas dengan $(1-\lambda_{n+1})$ didapatkan:

$$\lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_n f(x_n) \geq (1-\lambda_{n+1}) f\left(\frac{\lambda_1}{1-\lambda_{n+1}}x_1+\frac{\lambda_2}{1-\lambda_{n+1}}x_2+\cdots+\frac{\lambda_n}{1-\lambda_{n+1}}x_n\right)$$

Tambahkan kedua ruas dengan $\lambda_{n+1} f(x_{n+1})$:

$$\lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_n f(x_n)+\lambda_{n+1} f(x_{n+1}) \geq (1-\lambda_{n+1}) f\left(\frac{\lambda_1}{1-\lambda_{n+1}}x_1+\frac{\lambda_2}{1-\lambda_{n+1}}x_2+\cdots+\frac{\lambda_n}{1-\lambda_{n+1}}x_n\right)+\lambda_{n+1} f(x_{n+1})$$

atau $$\sum_{i=1}^{n+1}\lambda_i f(x_i) \geq (1-\lambda_{n+1})\left(f\left(\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}x_i\right)\right)+\lambda_{n+1} f(x_{n+1})$$

Karena $(1-\lambda_{n+1})+\lambda_{n+1}=1$ dan f adalah fungsi konveks, maka

berdasarkan definisi fungsi konveks, didapat:

$$\begin{aligned}
\sum_{i=1}^{n+1}\lambda_i f(x_i) &\geq (1-\lambda_{n+1}) f\left(\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}x_i\right)+\lambda_{n+1} f(x_{n+1}) \\
&\geq f\left((1-\lambda_{n+1})\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}x_i+\lambda_{n+1}x_{n+1}\right) \\
&= f\left(\sum_{i=1}^{n}(1-\lambda_{n+1})\frac{\lambda_i}{1-\lambda_{n+1}}x_i+\lambda_{n+1}x_{n+1}\right) \\
&= f\left(\sum_{i=1}^{n}\lambda_i x_i+\lambda_{n+1}x_{n+1}\right) = f\left(\sum_{i=1}^{n+1}\lambda_i x_i\right)
\end{aligned}$$

Jadi:

$$\sum_{i=1}^{n+1} \lambda_i f(x_i) \quad \geq \quad f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right)$$

Jadi, terbukti bahwa untuk (n+1) pertidaksamaan Jensen benar.

∴ Karena untuk n = 2, n, dan (n+1) pertidaksamaan Jensen terbukti benar, maka dapat disimpulkan pertidaksamaan Jensen benar untuk setiap n.

"$\forall n$, jika *f* adalah fungsi konveks, dan untuk $\lambda_i \in [0,1]$ sedemikian sehingga $\sum_{i=1}^{n} \lambda_i = 1$, maka berlaku $\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$ "

LAMPIRAN 4

Pembuktian:

Jika *u*(*x*) adalah sebarang fungsi dari variabel random *X* dan *u*(*x*) adalah fungsi konveks, maka berlaku:

$$E[u(x)] \geq u(E[X])$$

Bukti:

Misalkan *X* adalah variabel random diskret dan *f*(*x*) adalah pdf dari *X*.

Misalkan terdapat *n* observasi, maka $\sum_{i=1}^{n} f(x_i) = 1$.

Dan:

$$E[u(x)] = \sum_{i=1}^{n} u(x_i).f(x_i)$$

Jika *u*(*x*) adalah fungsi konveks, maka berdasarkan pertidaksamaan Jensen, $E\left[u\left(x\right)\right]$ dapat dituliskan menjadi:

$$\begin{aligned} E\left[u\left(x\right)\right] &= \sum_{i=1}^{n} u(x_i).f(x_i) \\ &\geq u\left(\sum_{i=1}^{n} x_i.f(x_i)\right) \\ &= u\left(E\left[x_i\right]\right) \end{aligned}$$

Jadi: $E\left[u\left(x\right)\right] \geq u\left(E\left[x_i\right]\right)$.

$\therefore$ Terbukti bahwa Jika *u*(*x*) adalah sebarang fungsi dari variabel random *X* dan *u*(*x*) adalah fungsi konveks, maka berlaku:

$$E\left[u\left(x\right)\right] \geq u\left(E\left[X\right]\right)$$

LAMPIRAN 5

Menunjukkan: $\frac{\partial^2 l}{\partial {\pi_m}^2} < 0$, $\forall \pi_m$ dan $\frac{\partial^2 l}{\partial {P_{k|m}}^2} < 0$, $\forall P_{k|m}$.

Bukti:

- $\frac{\partial^2 l}{\partial {\pi_m}^2} < 0$, $\forall \pi_m$

Diketahui bahwa: $\frac{\partial l}{\partial \pi_m} = \sum_{i=1}^{n} \frac{\prod_{k=1}^{K} {P_{k|m}}^{y_{ki}} \left(1 - P_{k|m}\right)^{1-y_{ki}}}{\sum_{s=1}^{M} \pi_s \prod_{j=1}^{K} {P_{j|s}}^{y_{ji}} \left(1 - P_{j|s}\right)^{1-y_{ji}}}$ untuk $m = 1, 2, \ldots, M$.

Atau:

$$\frac{\partial l}{\partial \pi_m} = \sum_{i=1}^{n} \frac{\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}}}{\pi_m \prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} + \sum_{\substack{s=1 \\ s\neq m}}^{M} \pi_s \prod_{k=1}^{K} P_{k|s}{}^{y_{ki}} \left(1-P_{k|s}\right)^{1-y_{ki}}}$$

$$= \sum_{i=1}^{n} \left\{ \prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} . \left[\pi_m \prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} + \sum_{\substack{s=1 \\ s\neq m}}^{M} \pi_s \prod_{k=1}^{K} P_{k|s}{}^{y_{ki}} \left(1-P_{k|s}\right)^{1-y_{ki}} \right]^{-1} \right\}$$

Maka:

$$\frac{\partial^2 l}{\partial \pi_m{}^2} = \sum_{i=1}^{n} \left\{ \begin{array}{l} (-1) \times \left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right) \times \left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right) \\ \times \left[\sum_{s=1}^{M} \pi_s \prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right]^{-2} \end{array} \right\}$$

$$= \sum_{i=1}^{n} \left\{ (-1) \frac{\left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right) \left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right)}{\left[\sum_{s=1}^{M} \pi_s \prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right]^{2}} \right\}$$

$$= -\sum_{i=1}^{n} \left\{ \frac{\left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right)^{2}}{\left[\sum_{s=1}^{M} \pi_s \prod_{k=1}^{K} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \right]^{2}} \right\}$$

Karena $\dfrac{\left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}\right)^2}{\left[\sum_{s=1}^{M}\pi_s\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}\right]^2}$ pasti selalu positif, maka

$-\sum_{i=1}^{n}\left\{\dfrac{\left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}\right)^2}{\left[\sum_{s=1}^{M}\pi_s\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}\right]^2}\right\}$ pasti bernilai negatif.

Dengan perkataan lain, $-\sum_{i=1}^{n}\left\{\dfrac{\left(\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}\right)^2}{\left[\sum_{s=1}^{M}\pi_s\prod_{k=1}^{K} P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}\right]^2}\right\}<0$.

$\therefore$ jadi, terbukti bahwa $\dfrac{\partial^2 l}{\partial \pi_m{}^2}<0$, $\forall \pi_m$

- $\dfrac{\partial^2 l}{\partial P_{k|m}{}^2}<0$, $\forall P_{k|m}$

Diketahui:

$$\frac{\partial l}{\partial P_{km}}=\sum_{i=1}^{n}\left\{\frac{\left[y_{ki}P_{k|m}{}^{y_{ki}-1}\left(1-P_{k|m}\right)^{1-y_{ki}}-\left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}}P_{k|m}{}^{y_{ki}}\right]\left[\pi_m\prod_{\substack{j=1\\ j\neq k}}^{K}P_{j|m}{}^{y_{ji}}\left(1-P_{j|m}\right)^{1-y_{ji}}\right]}{\sum_{s=1}^{M}\pi_s\prod_{j=1}^{K}P_{j|s}{}^{y_{ji}}\left(1-P_{j|s}\right)^{1-y_{ji}}}\right\}$$

untuk $m=1,2,\ldots,M$; k = 1,2,…,K.

Definisikan:

$$U=\left[\frac{y_{ki}}{P_{k|m}}-\frac{(1-y_{ki})}{(1-P_{k|m})}\right]\left[\pi_m\prod_{j=1}^{K}P_{j|m}{}^{y_{ji}}\left(1-P_{j|m}\right)^{1-y_{ji}}\right]$$

dan

$$V=\sum_{s=1}^{M}\pi_s\prod_{j=1}^{K}P_{j|s}{}^{y_{ji}}\left(1-P_{j|s}\right)^{1-y_{ji}}$$

$\frac{\partial^2 l}{\partial P_{km}{}^2}$ dihitung dengan: $\frac{\partial^2 l}{\partial P_{km}{}^2}=\frac{U'V-V'U}{V^2}$

Oleh karena itu, perlu dicari *U'* dan *V'* terlebih dahulu.

- *U'*

$$U=\left[\frac{y_{ki}}{P_{k|m}}-\frac{(1-y_{ki})}{(1-P_{k|m})}\right]\left[\pi_m\prod_{j=1}^{K}P_{j|m}{}^{y_{ji}}\left(1-P_{j|m}\right)^{1-y_{ji}}\right]$$

$$=\left[\frac{y_{ki}}{P_{k|m}}P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}-\frac{(1-y_{ki})}{(1-P_{k|m})}P_{k|m}{}^{y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}}\right]\left[\pi_m\prod_{\substack{j=1\\j\neq k}}^{K}P_{j|m}{}^{y_{ji}}\left(1-P_{j|m}\right)^{1-y_{ji}}\right]$$

$$=\left[y_{ki}P_{k|m}{}^{y_{ki}-1}\left(1-P_{k|m}\right)^{1-y_{ki}}-(1-y_{ki})\left(1-P_{k|m}\right)^{-y_{ki}}P_{k|m}{}^{y_{ki}}\right]\left[\pi_m\prod_{\substack{j=1\\j\neq k}}^{K}P_{j|m}{}^{y_{ji}}\left(1-P_{j|m}\right)^{1-y_{ji}}\right]$$

Definisikan:

$U_1=y_{ki}P_{k|m}{}^{y_{ki}-1}$; $U_1'=y_{ki}\left(y_{ki}-1\right)P_{k|m}{}^{y_{ki}-2}$

$V_1=\left(1-P_{k|m}\right)^{1-y_{ki}}$; $V_1'=-(1-y_{ki})\left(1-P_{k|m}\right)^{-y_{ki}}$

$U_2=(1-y_{ki})\left(1-P_{k|m}\right)^{-y_{ki}}$; $U_2'=-y_{ki}\left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}-1}$

$V_2=P_{k|m}{}^{y_{ki}}$; $V_2'=y_{ki}P_{k|m}{}^{y_{ki}-1}$

Maka, *U'* didapatkan dengan:

$$U' = \left[\left(U_1'V_1 + U_1V_1'\right) - \left(U_2'V_2 + U_2V_2'\right)\right]\left[\pi_m \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}^{\ y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right]$$

Jadi,

$$U' = \begin{bmatrix} \left(y_{ki}\left(y_{ki}-1\right)P_{k|m}^{\ y_{ki}-2}\left(1-P_{k|m}\right)^{1-y_{ki}} + y_{ki}P_{k|m}^{\ y_{ki}-1}\left(-\left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}}\right)\right) \\ -\left(\left(-y_{ki}\left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}-1}\right)P_{k|m}^{\ y_{ki}} + \left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}} y_{ki}P_{k|m}^{\ y_{ki}-1}\right) \end{bmatrix}$$

$$\times\left[\pi_m \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}^{\ y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right]$$

$$= \begin{bmatrix} \left(y_{ki}\left(y_{ki}-1\right)P_{k|m}^{\ y_{ki}-2}\left(1-P_{k|m}\right)^{1-y_{ki}} + y_{ki}\left(y_{ki}-1\right)P_{k|m}^{\ y_{ki}-1}\left(1-P_{k|m}\right)^{-y_{ki}}\right) \\ -\left(y_{ki}\left(y_{ki}-1\right)\left(1-P_{k|m}\right)^{-y_{ki}-1}P_{k|m}^{\ y_{ki}} + y_{ki}\left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}} P_{k|m}^{\ y_{ki}-1}\right) \end{bmatrix}$$

$$\times\left[\pi_m \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}^{\ y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right]$$

Karena y_{ki} hanya bernilai 1 atau 0, untuk semua *k* dan *i* , maka

$y_{ki}\left(y_{ki}-1\right)=0$, $\forall y_{ki} = 0,1$. Oleh karena itu, *U'* = 0.

- *V'*

$$V = \sum_{s=1}^{M} \pi_s \prod_{j=1}^{K} P_{j|s}^{\ y_{ji}} \left(1-P_{j|s}\right)^{1-y_{ji}}$$

$$= \left(\pi_m P_{k|m}^{\ y_{ki}}\left(1-P_{k|m}\right)^{1-y_{ki}} \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}^{\ y_{ji}}\left(1-P_{j|m}\right)^{1-y_{ji}}\right) + \left(\sum_{\substack{s=1\\ s\neq m}}^{M} \pi_s \prod_{j=1}^{K} P_{j|s}^{\ y_{ji}}\left(1-P_{j|s}\right)^{1-y_{ji}}\right)$$

Definisikan:

$$U_3 = \pi_m P_{k|m}{}^{y_{ki}} \quad ; U_3' = \pi_m y_{ki} P_{k|m}{}^{y_{ki}-1}$$

$$V_3 = \left(1-P_{k|m}\right)^{1-y_{ki}} \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}} ; V_3' = -\left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}} \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}$$

maka, V' dapat dituliskan sebagai: $V' = U_3'V_3 + U_3V_3'$.

$$\begin{aligned}
V' &= \left(\pi_m y_{ki} P_{k|m}{}^{y_{ki}-1} \left(1-P_{k|m}\right)^{1-y_{ki}} \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}} \right) \\
&\quad + \left(\pi_m P_{k|m}{}^{y_{ki}} \left(-\left(1-y_{ki}\right)\left(1-P_{k|m}\right)^{-y_{ki}} \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}} \right) \right) \\
&= \left(\frac{\pi_m y_{ki}}{P_{k|m}} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}} \right) \\
&\quad - \left(\frac{\pi_m \left(1-y_{ki}\right)}{\left(1-P_{k|m}\right)} P_{k|m}{}^{y_{ki}} \left(1-P_{k|m}\right)^{1-y_{ki}} \prod_{\substack{j=1\\ j\neq k}}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}} \right) \\
&= \left(\frac{\pi_m y_{ki}}{P_{k|m}} \prod_{j=1}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}} \right) - \left(\frac{\pi_m \left(1-y_{ki}\right)}{\left(1-P_{k|m}\right)} \prod_{j=1}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}} \right) \\
&= \left(\frac{y_{ki}}{P_{k|m}} - \frac{\left(1-y_{ki}\right)}{\left(1-P_{k|m}\right)} \right) \pi_m \prod_{j=1}^{K} P_{j|m}{}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}
\end{aligned}$$

Jadi:

$$
\begin{aligned}
\frac{\partial^2 l}{\partial {P_{km}}^2} &= \frac{U'V - V'U}{V^2} \\
&= \frac{0 - V'U}{V^2} \qquad \leftarrow \text{ karena } U' = 0 \\
&= \frac{-\left(\left(\frac{y_{ki}}{P_{k|m}} - \frac{(y_{ki}-1)}{(1-P_{k|m})}\right)\pi_m \prod_{j=1}^{K} {P_{j|m}}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right)}{\left(\sum_{s=1}^{M} \pi_s \prod_{j=1}^{K} {P_{j|s}}^{y_{ji}} \left(1-P_{j|s}\right)^{1-y_{ji}}\right)^2} \\
&\quad \times \left(\left(\frac{y_{ki}}{P_{k|m}} - \frac{(1-y_{ki})}{(1-P_{k|m})}\right)\pi_m \prod_{j=1}^{K} {P_{j|m}}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right) \\
&= -\frac{\left(\left(\frac{y_{ki}}{P_{k|m}} - \frac{(y_{ki}-1)}{(1-P_{k|m})}\right)\pi_m \prod_{j=1}^{K} {P_{j|m}}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right)^2}{\left(\sum_{s=1}^{M} \pi_s \prod_{j=1}^{K} {P_{j|s}}^{y_{ji}} \left(1-P_{j|s}\right)^{1-y_{ji}}\right)^2}
\end{aligned}
$$

Karena $\frac{\left(\left(\frac{y_{ki}}{P_{k|m}} - \frac{(y_{ki}-1)}{(1-P_{k|m})}\right)\pi_m \prod_{j=1}^{K} {P_{j|m}}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right)^2}{\left(\sum_{s=1}^{M} \pi_s \prod_{j=1}^{K} {P_{j|s}}^{y_{ji}} \left(1-P_{j|s}\right)^{1-y_{ji}}\right)^2}$ selalu bernilai positif,

maka $-\frac{\left(\left(\frac{y_{ki}}{P_{k|m}} - \frac{(y_{ki}-1)}{(1-P_{k|m})}\right)\pi_m \prod_{j=1}^{K} {P_{j|m}}^{y_{ji}} \left(1-P_{j|m}\right)^{1-y_{ji}}\right)^2}{\left(\sum_{s=1}^{M} \pi_s \prod_{j=1}^{K} {P_{j|s}}^{y_{ji}} \left(1-P_{j|s}\right)^{1-y_{ji}}\right)^2}$ selalu bernilai negatif

(> 0).

$\therefore$ terbukti bahwa $\frac{\partial^2 l}{\partial {P_{km}}^2} < 0$.