LAMPIRAN

LAMPIRAN

LAMPIRAN 1

Bukti Lemma 2.18.

Misalkan matriks $\mathbf{A}$ berukuran $n \times n$ merupakan matriks definit positif. Lemma 2.21 akan dibuktikan dengan menggunakan kontradiksi. Andaikan $\mathbf{A}$ merupakan matriks yang *singular* atau secara ekuivalen, yaitu $rank(\mathbf{A}) < n$. Akibatnya, kolom $\mathbf{A}$ tidak bebas linier sehingga terdapat vektor tidak nol $\mathbf{x}_*$, sedemikian sehingga $\mathbf{A}\mathbf{x}_* = \mathbf{0}$. Oleh karena itu,

$$\mathbf{x}_*^T \mathbf{A}\mathbf{x}_* = \mathbf{x}_*^T (\mathbf{A}\mathbf{x}_*) = \mathbf{x}_*^T \mathbf{0} = 0 .$$

Padahal $\mathbf{A}$ merupakan matriks definit positif, artinya

$$\mathbf{x}^\mathbf{T} \mathbf{A}\mathbf{x} > 0, \text{ untuk setiap } \mathbf{x} \neq 0 .$$

Sedangkan, jika $\mathbf{A}$ merupakan matriks *singular* diperoleh bahwa:

$$\mathbf{x}_*^T \mathbf{A}\mathbf{x}_* = \mathbf{x}_*^T (\mathbf{A}\mathbf{x}_*) = \mathbf{x}_*^T \mathbf{0} = 0$$

Berarti kontradiksi dengan pengandaian yang digunakan sehingga sembarang matriks definit positif adalah *nonsingular*.

LAMPIRAN 2

Bukti (2.3.2.a), (2.3.2.b), dan (2.3.2.c).

Misalkan a_i dinyatakan sebagai entri ke-*i* dari **a** dan x_i sebagai entri ke-*i* dari **x**, sedangkan a_{ik} dinyatakan sebagai entri ke-*ik* dari **A**. Maka

$$\mathbf{a}^T\mathbf{x} = \sum_i a_i x_i \text{ dan } \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i,k} a_{ik} x_i x_k$$

Sebagai langkah awal akan didefinisikan:

$$\frac{\partial x_i}{\partial x_j} = \begin{cases} 1, & \text{jika } i = j \\ 0, & \text{jika } i \neq j \end{cases}$$

$$\frac{\partial (x_i, x_k)}{\partial x_j} = \begin{cases} 2x_j, & \text{jika } i = k = j \\ x_i, & \text{jika } k = j \text{ tetapi } i \neq j \\ x_k, & \text{jika } i = j \text{ tetapi } k \neq j \\ 0, & \text{lainnya.} \end{cases}$$

sehingga diperoleh

$$\frac{\partial(\mathbf{a}^T\mathbf{x})}{\partial x_j} = \frac{\partial\left(\sum_i a_i x_i\right)}{\partial x_j} = \sum_i a_i \frac{\partial x_i}{\partial x_j} = a_j \qquad \textbf{(L2.1)}$$

dan

$$\frac{\partial(\mathbf{x}'\mathbf{A}\mathbf{x})}{\partial x_j} = \frac{\partial\left(\sum_{i,k} a_{ik} x_i x_k\right)}{\partial x_j}$$

$$= \frac{\partial\left(a_{jj}x_j^2 + \sum_{i \neq j} a_{ij}x_i x_j + \sum_{k \neq j} a_{jk}x_j x_k + \sum_{i \neq j, k \neq j} a_{ik}x_i x_k\right)}{\partial x_j}$$

$$= a_{jj}\frac{\partial x_j^2}{\partial x_j} + \sum_{i \neq j} a_{ij}\frac{\partial\left(x_i x_j\right)}{\partial x_j} + \sum_{k \neq j} a_{jk}\frac{\partial\left(x_j x_k\right)}{\partial x_j} + \sum_{i \neq j, k \neq j} a_{ik}\frac{\partial\left(x_i x_k\right)}{\partial x_j}$$

$$= 2a_{jj}x_j + \sum_{i \neq j} a_{ij}x_i + \sum_{k \neq j} a_{jk}x_k + 0$$

$$= \sum_i a_{ij}x_i + \sum_k a_{jk}x_k. \qquad \textbf{(L2.2)}$$

Persamaan (L2.1) dapat dibentuk dalam notasi vektor sebagai

$$\frac{\partial\left(\mathbf{a}^T\,\mathbf{x}\right)}{\partial\,\mathbf{x}} = \mathbf{a} \quad \text{atau} \quad \frac{\partial\left(\mathbf{a}^T\,\mathbf{x}\right)}{\partial\,\mathbf{x}^T} = \mathbf{a}^T$$

Sedangkan persamaan (L2.2) dapat dibentuk dalam notasi matriks sebagai:

$$\frac{\partial\left(\mathbf{x}^T\,\mathbf{A}\,\mathbf{x}\right)}{\partial\,\mathbf{x}} = \left(\mathbf{A} + \mathbf{A}^T\right)\mathbf{x}$$

dengan $\sum_k a_{jk}x_k$ adalah entri ke-*j* dari vektor kolom $\mathbf{A}\,\mathbf{x}$ dan $\sum_i a_{ij}x_i$ adalah entri ke-*j* dari $\mathbf{A}^T\,\mathbf{x}$.

LAMPIRAN 3

Bukti (2.3.3.a).

Misalkan $\mathbf{L} = a\mathbf{F} + b\mathbf{G}$. Elemen ke-*is* dari **L** adalah sebagai berikut:

$$l_{is} = af_{is} + bg_{is}.$$

Fungsi f_{is} dan g_{is} *continuously differentiable* di $\mathbf{c}$ sehingga l_{is} juga *continuously differentiable* di $\mathbf{x} = \mathbf{c}$, yakni:

$$\frac{\partial l_{is}}{\partial x_j} = a\frac{\partial f_{is}}{\partial x_j} + b\frac{\partial g_{is}}{\partial x_j}.$$

Oleh karena $\partial l_{is}/\partial x_j$, $\partial f_{is}/\partial x_j$, dan $\partial g_{is}/\partial x_j$ masing-masing merupakan elemen ke-*is* dari $\partial \mathbf{L}/\partial x_j$, $\partial \mathbf{F}/\partial x_j$, dan $\partial \mathbf{G}/\partial x_j$, berarti **L** juga *continuously differentiable* di $\mathbf{x} = \mathbf{c}$, yaitu:

$$\frac{\partial \mathbf{L}}{\partial x_j} = a\frac{\partial \mathbf{F}}{\partial x_j} + b\frac{\partial \mathbf{G}}{\partial x_j}.$$

LAMPIRAN 4

Bukti (2.3.3.b).

Misalkan $\mathbf{H} = \mathbf{FG}$. Elemen ke-*it* dari $\mathbf{H}$ adalah sebagai berikut:

$$h_{it} = \sum_{s=1}^{q} f_{is} g_{st} .$$

Fungsi f_{is} dan g_{st} *continuously differentiable* di $\mathbf{c}$ sehingga $f_{is} g_{st}$ juga *continuously differentiable* di $\mathbf{x} = \mathbf{c}$, yakni:

$$\frac{\partial f_{is} g_{st}}{\partial x_j} = f_{is} \frac{\partial g_{st}}{\partial x_j} + \frac{\partial f_{is}}{\partial x_j} g_{st} .$$

Oleh karena itu, h_{it} *continuously differentiable* di $\mathbf{x} = \mathbf{c}$ atau dapat ditulis

$$\frac{\partial h_{it}}{\partial x_j} = \sum_{s=1}^{q} \frac{\partial \left(f_{is} g_{st} \right)}{\partial x_j} = \sum_{s=1}^{q} f_{is} \frac{\partial g_{st}}{\partial x_j} + \sum_{s=1}^{q} \frac{\partial f_{is}}{\partial x_j} g_{st} .$$

Oleh karena $\sum_{s=1}^{q} f_{is} \left(\partial g_{st} / \partial x_j \right)$ dan $\sum_{s=1}^{q} \left(\partial f_{is} / \partial x_j \right) g_{st}$ masing-masing adalah elemen ke-*it* dari $\mathbf{F} \left(\partial \mathbf{G} / \partial x_j \right)$ dan $\left(\partial \mathbf{F} / \partial x_j \right) \mathbf{G}$), dapat disimpulkan bahwa **H** juga *continuously differentiable* di $\mathbf{x} = \mathbf{c}$, yaitu:

$$\frac{\partial \mathbf{FG}}{\partial x_j} = \mathbf{F} \frac{\partial \mathbf{G}}{\partial x_j} + \frac{\partial \mathbf{F}}{\partial x_j} \mathbf{G} .$$

LAMPIRAN 5

Bukti (2.3.7.a), (2.3.7.b), dan (2.3.7.c).

Berdasarkan (2.3.3.b), (2.3.4.a), dan (2.3.6.c) didapatkan bahwa

$$\frac{\partial^2 \det(\mathbf{F})}{\partial x_i \partial x_j} = \frac{\partial\left\{|\mathbf{F}|\, tr\left(\mathbf{F}^{-1}\,\partial\mathbf{F}/\partial x_j\right)\right\}}{\partial x_i}$$

$$= |\mathbf{F}|\left[tr\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i \partial x_j}\right) + tr\left(-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\right)\right]$$

$$+ |\mathbf{F}|\, tr\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\right) tr\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\right)$$

$$= |\mathbf{F}|\begin{bmatrix} tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_i \partial x_j}\right) + tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_i}\right) tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_j}\right) \\ -tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_j}\right) \end{bmatrix}$$

dan dengan menggunakan (2.3.3.c) dan (2.3.6.e) didapatkan pula bahwa

$$\frac{\partial^2 \mathbf{F}^{-1}}{\partial x_i \partial x_j} = \frac{\partial\left[-\mathbf{F}^{-1}\left(\partial\mathbf{F}/\partial x_j\right)\mathbf{F}^{-1}\right]}{\partial x_i}$$

$$= -\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\frac{\partial\mathbf{F}^{-1}}{\partial x_i} - \mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i \partial x_j}\mathbf{F}^{-1} - \frac{\partial\mathbf{F}^{-1}}{\partial x_i}\frac{\partial\mathbf{F}}{\partial x_j}\mathbf{F}^{-1}$$

$$= -\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\left(-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\right) - \mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i \partial x_j}\mathbf{F}^{-1}$$

$$-\left(-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\right)\frac{\partial\mathbf{F}}{\partial x_j}\mathbf{F}^{-1}$$

$$= -\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i \partial x_j}\mathbf{F}^{-1} + \mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_j}\mathbf{F}^{-1}$$

$$+ \mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_j}\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i}\mathbf{F}^{-1}.$$

Selanjutnya, berdasarkan (2.3.4.a), (2.3.6.d), dan (2.3.6.e) didapatkan bahwa

$$\frac{\partial^2 \log\det(\mathbf{F})}{\partial x_i \partial x_j} = \frac{\partial tr\left[\mathbf{F}^{-1}\left(\partial \mathbf{F}/\partial x_j\right)\right]}{\partial x_i}$$

$$= tr\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i \partial x_j}\right) + tr\left(-\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_j}\right)$$

$$= tr\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i \partial x_j}\right) - tr\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_j}\right).$$

LAMPIRAN 6

Bukti $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta}) \leftrightarrow \boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$.

Keterangan: $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_\mathrm{n})$.

Bukti:

$(\rightarrow)$ Oleh karena $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta})$, maka:

- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$
- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$

  $\vdots$

- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_n} = 0,$
- $\dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \ i = 1, 2, \ldots, n,$
- $D = \dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} . \dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n \ , i \neq j$.

Akan ditunjukkan bahwa $\boldsymbol{\theta}$ juga memaksimumkan $\ln L(\boldsymbol{\theta})$, yaitu:

- $\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = \frac{1}{L(\boldsymbol{\theta})} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = \frac{1}{L(\boldsymbol{\theta})} .0 = 0$$

$$\therefore \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$$

- $\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = 0$

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = \frac{1}{L(\boldsymbol{\theta})} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = \frac{1}{L(\boldsymbol{\theta})} .0 = 0$$

$$\therefore \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$$

$$\vdots$$

- $\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_n} = 0$

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_n} = \frac{1}{L(\boldsymbol{\theta})} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_n} = \frac{1}{L(\boldsymbol{\theta})} .0 = 0$$

$$\therefore \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_n} = 0,$$

- $\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \ i = 1, 2, \ldots, n,$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2} = \frac{\partial}{\partial\theta_i}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_i}\right) = \frac{\partial}{\partial\theta_i}\left(\frac{1}{L(\boldsymbol{\theta})}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\right)$$

$$= \frac{\partial}{\partial\theta_i}\left(\frac{1}{L(\boldsymbol{\theta})}\right)\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i} + \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{1}{L(\boldsymbol{\theta})} = \frac{\partial}{\partial\theta_i}\left(\frac{1}{L(\boldsymbol{\theta})}\right)\cdot 0 + \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{1}{L(\boldsymbol{\theta})}$$

$$= \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{1}{L(\boldsymbol{\theta})} < 0\cdot\frac{1}{L(\boldsymbol{\theta})} = 0$$

$$\therefore \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2} < 0,\ i = 1,2,\ldots,n,$$

- $D = \dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2} - \left(\dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2 > 0,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,n\ ,i \neq j$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2} = \frac{\partial}{\partial\theta_i}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_i}\right) = \frac{\partial}{\partial\theta_i}\left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L(\boldsymbol{\theta})}\right), i = 1,2,\ldots,n$$

$$= \frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L^2(\boldsymbol{\theta})}$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2} = \frac{\partial}{\partial\theta_j}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_j}\right) = \frac{\partial}{\partial\theta_j}\left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L(\boldsymbol{\theta})}\right), j = 1,2,\ldots,n$$

$$= \frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L^2(\boldsymbol{\theta})}$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_j}\right) = \frac{\partial}{\partial\theta_i}\left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L(\boldsymbol{\theta})}\right), i = 1,2,\ldots,n,\ j = 1,2,\ldots,n$$

$$= \frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L^2(\boldsymbol{\theta})}.$$

$$D = \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2$$

$$= \left(\frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L^2(\boldsymbol{\theta})}\right)\cdot\left(\frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L^2(\boldsymbol{\theta})}\right)$$

$$-\left(\frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L^2(\boldsymbol{\theta})}\right)^2, i = 1,2,\cdots,n,\ j = 1,2,\cdots,n$$

$$= \frac{1}{L^4(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot L^2(\boldsymbol{\theta}) + \left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\right.$$

$$\left. - \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\cdot L(\boldsymbol{\theta}) - \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\right)^2\cdot L(\boldsymbol{\theta})\right]$$

$$- \frac{1}{L^4(\boldsymbol{\theta})}\left[\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\cdot L^2(\boldsymbol{\theta}) + \left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\right.$$

$$\left. -2\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot L(\boldsymbol{\theta})\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)\right]$$

$$= \frac{L^2(\boldsymbol{\theta})}{L^4(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\right]$$

$$+\frac{L(\boldsymbol{\theta})}{L^4(\boldsymbol{\theta})}\left[2\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\right.$$

$$\left.-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\right)^2\right]$$

$$=\frac{1}{L^2(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\right]$$

$$+\frac{1}{L^3(\boldsymbol{\theta})}\left[2\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot 0\cdot 0-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot(0)^2-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot(0)^2\right]$$

$$=\frac{1}{L^2(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\right]>\frac{1}{L^2(\boldsymbol{\theta})}\cdot 0=0$$

$$\therefore D=\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2}.\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2>0,\ i=1,2,\ldots,n,\ j=1,2,\ldots,n\ ,i\neq j.$$

Dengan perkataan lain terbukti bahwa $\boldsymbol{\theta}$ juga memaksimumkan $\ln L(\boldsymbol{\theta})$.

$(\leftarrow)$ Oleh karena $\boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$, maka

- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_1}=0,$
- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_2}=0,$

  $\vdots$

- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_n} = 0,$

- $\dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \; i = 1, 2, \ldots, n,$

- $D = \dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} . \dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n \; , i \neq j.$

Akan ditunjukkan bahwa $\boldsymbol{\theta}$ juga memaksimumkan $L(\boldsymbol{\theta})$, yaitu:

- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = L(\boldsymbol{\theta}) \cdot \frac{1}{L(\boldsymbol{\theta})} . \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = L(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$$

- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = L(\boldsymbol{\theta}) \cdot \frac{1}{L(\boldsymbol{\theta})} . \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = L(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$$

$$\vdots$$

- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_n} = 0,$

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_n} = L(\boldsymbol{\theta}) \cdot \frac{1}{L(\boldsymbol{\theta})} . \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_n} = L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_n} = L(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_n} = 0,$$

- $\dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \ i = 1, 2, \ldots, n,$

$$\begin{aligned}
\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} &= \frac{\partial \left(\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)}{\partial \theta_i} \\
&= \frac{\partial}{\partial \theta_i} \left(L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_i} \right) \\
&= \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_i} + L(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \\
&= \frac{1}{L(\boldsymbol{\theta})} \cdot \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \\
&= \frac{1}{L(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \right] \\
&= \frac{1}{L(\boldsymbol{\theta})} \cdot \left[(0)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \right] \\
&= L(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} < L(\boldsymbol{\theta}) \cdot 0 = 0
\end{aligned}$$

$$\therefore \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \ i = 1, 2, \ldots, n,$$

- $D = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} . \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n \;, i \neq j$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} = \frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2}}{L(\boldsymbol{\theta})}, i = 1, 2, \ldots, n$$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} = \frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2}}{L(\boldsymbol{\theta})}, \; j = 1, 2, \ldots, n$$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} = \frac{\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}}{L(\boldsymbol{\theta})}, i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n$$

$$\begin{aligned} D &= \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 \\ &= \left(\frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2}}{L(\boldsymbol{\theta})} \right) \cdot \left(\frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2}}{L(\boldsymbol{\theta})} \right) \\ &\quad - \left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}}{L(\boldsymbol{\theta})} \right)^2, i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n, i \neq j \end{aligned}$$

$$= \frac{1}{L^2(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^4(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + \right.$$

$$\left. + L^2(\boldsymbol{\theta}) \cdot \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + L^2(\boldsymbol{\theta}) \cdot \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \right]$$

$$- \frac{1}{L^2(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^4(\boldsymbol{\theta}) \cdot \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 + 2 \cdot L^2(\boldsymbol{\theta}) \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right]$$

$$= \frac{L^4(\boldsymbol{\theta})}{L^2(\boldsymbol{\theta})} \cdot \left[\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 \right]$$

$$+ \frac{L^2(\boldsymbol{\theta})}{L^2(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \right.$$

$$\left. - 2 \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right]$$

$$= L^2(\boldsymbol{\theta}) \cdot \left[\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 \right]$$

$$+ 1 \cdot \left[(0)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + (0)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} - 2 \cdot 0 \cdot 0 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right]$$

$$= L^2(\boldsymbol{\theta}) \cdot \left[\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 \right] > L^2(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore D = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} . \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n \;, i \neq j$$

Dengan perkataan lain, terbukti bahwa $\boldsymbol{\theta}$ juga memaksimumkan $L(\boldsymbol{\theta})$.

Terbukti bahwa $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta}) \leftrightarrow \boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$,

$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$.

LAMPIRAN 7

Metode Newton-Raphson.

Misalkan δ adalah vektor dari q-buah parameter yang tidak diketahui, yaitu $\delta = \left(\delta_0, \delta_1, \ldots, \delta_q\right)^T$, $l(\delta)$ adalah fungsi log-*likelihood* dari vektor δ. $\mathbf{s}(\delta)$ adalah vektor turunan parsial pertama dari $l(\delta)$ dengan elemen

$s_j(\delta) = \partial l(\delta) / \partial \delta_j,\ j = 0,1,\ldots,q$, yaitu $\mathbf{s}(\delta) = \begin{pmatrix} s_0(\delta) \\ s_1(\delta) \\ \vdots \\ s_q(\delta) \end{pmatrix} = \begin{pmatrix} \partial l(\delta) / \partial \delta_0 \\ \partial l(\delta) / \partial \delta_1 \\ \vdots \\ \partial l(\delta) / \partial \delta_q \end{pmatrix}$, sedangkan

$\mathbf{H}(\delta)$ menyatakan matriks turunan parsial kedua dari $l(\delta)$ dengan elemen

$$H_{jk}(\delta) = \frac{\partial^2 l(\delta)}{\partial \delta_k \partial \delta_j};\ j,k = 0,1,\ldots,q.$$

$$\mathbf{H}(\delta) = \frac{\partial \mathbf{s}(\delta)}{\partial \delta^T} = \begin{pmatrix} \frac{\partial \mathbf{s}(\delta)}{\partial \delta_0} & \frac{\partial \mathbf{s}(\delta)}{\partial \delta_1} & \cdots & \frac{\partial \mathbf{s}(\delta)}{\partial \delta_q} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial s_0(\delta)}{\partial \delta_0} & \frac{\partial s_0(\delta)}{\partial \delta_1} & \cdots & \frac{\partial s_0(\delta)}{\partial \delta_q} \\ \frac{\partial s_1(\delta)}{\partial \delta_0} & \frac{\partial s_1(\delta)}{\partial \delta_1} & \cdots & \frac{\partial s_1(\delta)}{\partial \delta_q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial s_q(\delta)}{\partial \delta_0} & \frac{\partial s_q(\delta)}{\partial \delta_1} & \cdots & \frac{\partial s_q(\delta)}{\partial \delta_q} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 l(\delta)}{\partial \delta_0^2} & \frac{\partial^2 l(\delta)}{\partial \delta_1 \partial \delta_0} & \cdots & \frac{\partial^2 l(\delta)}{\partial \delta_q \partial \delta_0} \\ \frac{\partial^2 l(\delta)}{\partial \delta_0 \partial \delta_1} & \frac{\partial^2 l(\delta)}{\partial \delta_1^2} & \cdots & \frac{\partial^2 l(\delta)}{\partial \delta_q \partial \delta_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 l(\delta)}{\partial \delta_0 \partial \delta_q} & \frac{\partial^2 l(\delta)}{\partial \delta_1 \partial \delta_q} & \cdots & \frac{\partial^2 l(\delta)}{\partial \delta_q^2} \end{pmatrix}$$

$$= \begin{pmatrix} H_{00} & H_{01} & \cdots & H_{0q} \\ H_{10} & H_{11} & \cdots & H_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ H_{q0} & H_{q1} & \cdots & H_{qq} \end{pmatrix}$$

Dengan menggunakan Metode Newton-Raphson akan dicari taksiran dari $\boldsymbol{\delta}$ yang merupakan penyelesaian dari $\mathbf{s}(\boldsymbol{\delta}) = \mathbf{0}$. Jika diberikan taksiran awal dari $\boldsymbol{\delta}$, yaitu $\hat{\boldsymbol{\delta}}^{(0)}$, maka dapat diperoleh pendekatan Deret Taylor Orde Pertama dari vektor $\mathbf{s}(\boldsymbol{\delta})$ sekitar $\boldsymbol{\delta} = \hat{\boldsymbol{\delta}}^{(0)}$, yaitu:

$$\mathbf{s}(\boldsymbol{\delta}) \approx \mathbf{s}\left(\hat{\boldsymbol{\delta}}^{(0)}\right) + \left.\frac{\partial \mathbf{s}(\boldsymbol{\delta})}{\partial \delta_0}\right|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} \left(\delta_0 - \hat{\delta}_0^{(0)}\right) + \left.\frac{\partial \mathbf{s}(\boldsymbol{\delta})}{\partial \delta_1}\right|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} \left(\delta_1 - \hat{\delta}_1^{(0)}\right) + \ldots + \left.\frac{\partial \mathbf{s}(\boldsymbol{\delta})}{\partial \delta_q}\right|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} \left(\delta_q - \hat{\delta}_q^{(0)}\right)$$

$$\approx \mathbf{s}\left(\hat{\boldsymbol{\delta}}^{(0)}\right) + \begin{pmatrix} \left.\frac{\partial \mathbf{s}(\boldsymbol{\delta})}{\partial \delta_0}\right|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} & \left.\frac{\partial \mathbf{s}(\boldsymbol{\delta})}{\partial \delta_1}\right|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} & \cdots & \left.\frac{\partial \mathbf{s}(\boldsymbol{\delta})}{\partial \delta_q}\right|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} \end{pmatrix} \begin{pmatrix} \left(\delta_0 - \hat{\delta}_0^{(0)}\right) \\ \left(\delta_1 - \hat{\delta}_1^{(0)}\right) \\ \vdots \\ \left(\delta_q - \hat{\delta}_q^{(0)}\right) \end{pmatrix}$$

$$\approx \mathbf{s}\left(\hat{\boldsymbol{\delta}}^{(0)}\right) + \begin{pmatrix} \frac{\partial s_0(\boldsymbol{\delta})}{\partial \delta_0} & \frac{\partial s_0(\boldsymbol{\delta})}{\partial \delta_1} & \cdots & \frac{\partial s_0(\boldsymbol{\delta})}{\partial \delta_q} \\ \frac{\partial s_1(\boldsymbol{\delta})}{\partial \delta_0} & \frac{\partial s_1(\boldsymbol{\delta})}{\partial \delta_1} & \cdots & \frac{\partial s_1(\boldsymbol{\delta})}{\partial \delta_q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial s_q(\boldsymbol{\delta})}{\partial \delta_0} & \frac{\partial s_q(\boldsymbol{\delta})}{\partial \delta_1} & \cdots & \frac{\partial s_q(\boldsymbol{\delta})}{\partial \delta_q} \end{pmatrix}_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} \left(\boldsymbol{\delta} - \hat{\boldsymbol{\delta}}^{(0)}\right)$$

$$\approx \mathbf{s}\left(\hat{\boldsymbol{\delta}}^{(0)}\right)+\begin{pmatrix} \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_0^2} & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_1 \partial \delta_0} & \cdots & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_q \partial \delta_0} \\ \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_0 \partial \delta_1} & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_1^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_q \partial \delta_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_0 \partial \delta_q} & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_1 \partial \delta_q} & \cdots & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_q^2} \end{pmatrix}_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} \left(\boldsymbol{\delta}-\hat{\boldsymbol{\delta}}^{(0)}\right)$$

$$\approx \mathbf{s}\left(\hat{\boldsymbol{\delta}}^{(0)}\right)+\begin{pmatrix} H_{00}(\boldsymbol{\delta}) & H_{01}(\boldsymbol{\delta}) & \cdots & H_{0q}(\boldsymbol{\delta}) \\ H_{10}(\boldsymbol{\delta}) & H_{11}(\boldsymbol{\delta}) & \cdots & H_{1q}(\boldsymbol{\delta}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{q0}(\boldsymbol{\delta}) & H_{q1}(\boldsymbol{\delta}) & \cdots & H_{qq}(\boldsymbol{\delta}) \end{pmatrix}_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}} \left(\boldsymbol{\delta}-\hat{\boldsymbol{\delta}}^{(0)}\right)$$

atau

$$\mathbf{s}(\boldsymbol{\delta}) \approx \mathbf{s}\left(\hat{\boldsymbol{\delta}}^{(0)}\right)+\mathbf{H}\left(\hat{\boldsymbol{\delta}}^{(0)}\right)\left(\boldsymbol{\delta}-\hat{\boldsymbol{\delta}}^{(0)}\right)$$

dengan

$\mathbf{s}\left(\hat{\boldsymbol{\delta}}^{(0)}\right)$ adalah vektor turunan parsial pertama dari $l(\boldsymbol{\delta})$ dengan elemen $s_j(\boldsymbol{\delta}) = \partial l(\boldsymbol{\delta}) / \partial \delta_j$ dihitung pada $\boldsymbol{\delta} = \hat{\boldsymbol{\delta}}^{(0)}$.

$\mathbf{H}\left(\hat{\boldsymbol{\delta}}^{(0)}\right) = \left.\frac{\partial \mathbf{s}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}^T}\right|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}^{(0)}}$ adalah matriks turunan parsial kedua dari $l(\boldsymbol{\delta})$ dengan elemen $H_{jk}(\boldsymbol{\delta}) = \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \delta_k \partial \delta_j}$ dihitung pada $\boldsymbol{\delta} = \hat{\boldsymbol{\delta}}^{(0)}$.

Oleh karena ingin dicari penyelesaian dari $\mathbf{s}(\boldsymbol{\delta}) = \mathbf{0}$, maka dapat ditulis

$$\mathbf{0} = \mathbf{s}(\delta) \approx \mathbf{s}\left(\hat{\delta}^{(0)}\right) + \mathbf{H}\left(\hat{\delta}^{(0)}\right)\left(\delta - \hat{\delta}^{(0)}\right)$$
$$\mathbf{H}\left(\hat{\delta}^{(0)}\right)\left(\delta - \hat{\delta}^{(0)}\right) = -\mathbf{s}\left(\hat{\delta}^{(0)}\right).$$

Jika matriks $\mathbf{H}\left(\hat{\delta}^{(0)}\right)$ *nonsingular* maka

$$\left(\mathbf{H}\left(\hat{\delta}^{(0)}\right)\right)^{-1}\mathbf{H}\left(\hat{\delta}^{(0)}\right)\left(\delta - \hat{\delta}^{(0)}\right) = -\left(\mathbf{H}\left(\hat{\delta}^{(0)}\right)\right)^{-1}\mathbf{s}\left(\hat{\delta}^{(0)}\right)$$
$$\left(\delta - \hat{\delta}^{(0)}\right) = -\left(\mathbf{H}\left(\hat{\delta}^{(0)}\right)\right)^{-1}\mathbf{s}\left(\hat{\delta}^{(0)}\right)$$
$$\delta = \hat{\delta}^{(0)} - \left(\mathbf{H}\left(\hat{\delta}^{(0)}\right)\right)^{-1}\mathbf{s}\left(\hat{\delta}^{(0)}\right).$$

Nilai δ menjadi taksiran baru yang kemudian dinotasikan dengan $\hat{\delta}^{(1)}$, yaitu:

$$\hat{\delta}^{(1)} = \hat{\delta}^{(0)} - \left(\mathbf{H}\left(\hat{\delta}^{(0)}\right)\right)^{-1}\mathbf{s}\left(\hat{\delta}^{(0)}\right)$$

Jika $\hat{\delta}^{(1)} \approx \hat{\delta}^{(0)}$ (misalkan $\left\|\hat{\delta}^{(1)} - \hat{\delta}^{(0)}\right\| < 10^{-5}$) maka hentikan proses iterasi dan kemudian ambil $\hat{\delta}^{(1)}$ sebagai taksiran δ. Sebaliknya, lanjutkan iterasi.

Jika diberikan $\hat{\delta}^{(1)}$, maka dapat diperoleh pendekatan Deret Taylor Orde Pertama dari vektor $\mathbf{s}(\delta)$ sekitar $\delta = \hat{\delta}^{(1)}$, yaitu:

$$\mathbf{s}(\delta) \approx \mathbf{s}\left(\hat{\delta}^{(1)}\right) + \mathbf{H}\left(\hat{\delta}^{(1)}\right)\left(\delta - \hat{\delta}^{(1)}\right)$$

dengan

$\mathbf{s}\left(\hat{\delta}^{(1)}\right)$ adalah vektor turunan parsial pertama dari $l(\delta)$ dengan elemen $s_j(\delta) = \partial l(\delta)/\partial \delta_j$ dihitung pada $\delta = \hat{\delta}^{(1)}$.

$\mathbf{H}\left(\hat{\delta}^{(1)}\right) = \left.\frac{\partial \mathbf{s}(\delta)}{\partial \delta^T}\right|_{\delta=\hat{\delta}^{(1)}}$ adalah matriks turunan parsial kedua dari $l(\delta)$ dengan elemen $H_{jk}(\delta) = \frac{\partial^2 l(\delta)}{\partial \delta_k \partial \delta_j}$ dihitung pada $\delta = \hat{\delta}^{(1)}$.

Dengan menyelesaikan $\mathbf{0} = \mathbf{s}(\delta) \approx \mathbf{s}\left(\hat{\delta}^{(1)}\right) + \mathbf{H}\left(\hat{\delta}^{(1)}\right)\left(\delta - \hat{\delta}^{(1)}\right)$

diperoleh $\mathbf{H}\left(\hat{\delta}^{(1)}\right)\left(\delta - \hat{\delta}^{(1)}\right) = -\mathbf{s}\left(\hat{\delta}^{(1)}\right)$.

Jika matriks $\mathbf{H}\left(\hat{\delta}^{(1)}\right)$ *nonsingular* maka

$$\left(\mathbf{H}\left(\hat{\delta}^{(1)}\right)\right)^{-1}\mathbf{H}\left(\hat{\delta}^{(1)}\right)\left(\delta - \hat{\delta}^{(1)}\right) = -\left(\mathbf{H}\left(\hat{\delta}^{(1)}\right)\right)^{-1}\cdot\mathbf{s}\left(\hat{\delta}^{(1)}\right)$$
$$\left(\delta - \hat{\delta}^{(1)}\right) = -\left(\mathbf{H}\left(\hat{\delta}^{(1)}\right)\right)^{-1}\cdot\mathbf{s}\left(\hat{\delta}^{(1)}\right)$$
$$\delta = \hat{\delta}^{(1)} - \left(\mathbf{H}\left(\hat{\delta}^{(1)}\right)\right)^{-1}\cdot\mathbf{s}\left(\hat{\delta}^{(1)}\right).$$

Nilai δ menjadi taksiran baru yang kemudian dinotasikan dengan $\hat{\delta}^{(2)}$, yaitu:

$$\hat{\delta}^{(2)} = \hat{\delta}^{(1)} - \left(\mathbf{H}\left(\hat{\delta}^{(1)}\right)\right)^{-1}\cdot\mathbf{s}\left(\hat{\delta}^{(1)}\right)$$

Dengan mengulang langkah yang sama dapat diperoleh taksiran $\hat{\delta}^{(3)}$, $\hat{\delta}^{(4)}$, dan seterusnya. Secara umum dapat ditentukan taksiran dari δ pada iterasi ke-$m+1$, yaitu $\hat{\delta}^{(m+1)}$, secara iteratif menggunakan formula:

$$\hat{\delta}^{(m+1)} = \hat{\delta}^{(m)} - \left(\mathbf{H}\left(\hat{\delta}^{(m)}\right)\right)^{-1} \cdot \mathbf{s}\left(\hat{\delta}^{(m)}\right); m = 0,1,\ldots$$

dengan

$\mathbf{s}\left(\hat{\delta}^{(m)}\right)$ adalah vektor turunan parsial pertama dari $l(\delta)$ dengan elemen $s_j(\delta) = \partial l(\delta) / \partial \delta_j$ dihitung pada $\delta = \hat{\delta}^{(m)}$.

$\mathbf{H}\left(\hat{\delta}^{(m)}\right)$ adalah matriks turunan parsial kedua dari $l(\delta)$ dengan elemen $H_{jk}(\delta) = \dfrac{\partial^2 l(\delta)}{\partial \delta_k \partial \delta_j}$ dihitung pada $\delta = \hat{\delta}^{(m)}$.

Hentikan proses iterasi jika $\hat{\delta}^{(m+1)} \approx \hat{\delta}^{(m)}$ (misalkan $\left\|\hat{\delta}^{(m+1)} - \hat{\delta}^{(m)}\right\| < 10^{-5}$), lalu ambil $\hat{\delta}^{(m+1)}$ sebagai taksiran dari δ.

LAMPIRAN 8

Bukti (2.7.3.b).

$(\rightarrow)$ Diketahui $T(\mathbf{y}) = c + \boldsymbol{l}^T\mathbf{y}$ dan $E(T(\mathbf{y})) = E(\lambda^T\alpha + \omega^T\mathbf{b})$ maka

$$\begin{aligned}
E(T(\mathbf{y})) &= E(c + \boldsymbol{l}^T\mathbf{y}) \\
&= E(c) + E(\boldsymbol{l}^T(\mathbf{X}\alpha + \mathbf{Z}\mathbf{b} + \mathbf{e})) \\
&= c + E(\boldsymbol{l}^T\mathbf{X}\alpha) + E(\boldsymbol{l}^T\mathbf{Z}\mathbf{b}) + E(\boldsymbol{l}^T\mathbf{e}) \\
&= c + \boldsymbol{l}^T\mathbf{X}\alpha + \boldsymbol{l}^T\mathbf{Z}E(\mathbf{b}) + \boldsymbol{l}^T E(\mathbf{e}) \\
&= c + \boldsymbol{l}^T\mathbf{X}\alpha \qquad \text{[karena } E(\mathbf{b}) = \mathbf{0} \text{ dan } E(\mathbf{e}) = \mathbf{0}\text{]} \qquad \textbf{(L8.1)}
\end{aligned}$$

sedangkan

$$\begin{aligned}
E(\lambda^T\alpha + \omega^T\mathbf{b}) &= E(\lambda^T\alpha) + E(\omega^T\mathbf{b}) \\
&= \lambda^T\alpha + \omega^T E(\mathbf{b}) \\
&= \lambda^T\alpha \qquad \text{[karena } E(\mathbf{b}) = \mathbf{0}\text{].} \qquad \textbf{(L8.2)}
\end{aligned}$$

Oleh karena $T(\mathbf{y})$ *unbiased* terhadap $\lambda^T\alpha + \omega^T\beta$, terlihat bahwa persamaan (L8.1) akan sama dengan (L8.2) jika $c = 0$ dan $\lambda^T = \boldsymbol{l}^T\mathbf{X}$.

$(\leftarrow)$ Diketahui bahwa $c = 0$ dan $\lambda^T = \boldsymbol{l}^T\mathbf{X}$ maka

$$\begin{aligned}
T(\mathbf{y}) &= c + \boldsymbol{l}^T\mathbf{y} \\
&= \boldsymbol{l}^T(\mathbf{X}\alpha + \mathbf{Z}\mathbf{b} + \mathbf{e}) \\
&= \boldsymbol{l}^T\mathbf{X}\alpha + \boldsymbol{l}^T\mathbf{Z}\mathbf{b} + \boldsymbol{l}^T\mathbf{e}.
\end{aligned}$$

Berikut ini akan dibuktikan bahwa $T(\mathbf{y})$ dapat mengestimasi kombinasi linier

$\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$

$$\begin{aligned} E\left(T(\mathbf{y})\right) &= E\left(\boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha}+\boldsymbol{l}^T\mathbf{Z}\mathbf{b}+\boldsymbol{l}^T\mathbf{e}\right) \\ &= E\left(\boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha}\right)+E\left(\boldsymbol{l}^T\mathbf{Z}\mathbf{b}\right)+E\left(\boldsymbol{l}^T\mathbf{e}\right) \\ &= \boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha}+\boldsymbol{l}^T\mathbf{Z}E(\mathbf{b})+\boldsymbol{l}^T E(\mathbf{e}) \\ &= \boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha} \quad \text{[karena } E(\mathbf{b})=\mathbf{0} \text{ dan } E(\mathbf{e})=\mathbf{0}\text{]} \\ &= \boldsymbol{\lambda}^T\boldsymbol{\alpha} \quad \text{[karena } \boldsymbol{\lambda}^T=\boldsymbol{l}^T\mathbf{X}\text{].} \end{aligned}$$

Jadi, terbukti bahwa $T(\mathbf{y})$ *unbiased* terhadap kombinasi linier $\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$.

Selanjutnya, akan dibuktikan bahwa $T(\mathbf{y})$ merupakan penaksir yang linier, yaitu: $T(a\mathbf{y}_1+b\mathbf{y}_2)=aT(\mathbf{y}_1)+bT(\mathbf{y}_2)$ dengan $T(\mathbf{y})=\boldsymbol{l}^T\mathbf{y}$.

$$\begin{aligned} T(a\mathbf{y}_1+b\mathbf{y}_2) &= \boldsymbol{l}^T(a\mathbf{y}_1+b\mathbf{y}_2) \\ &= \boldsymbol{l}^T a\mathbf{y}_1+\boldsymbol{l}^T b\mathbf{y}_2 \\ &= a\left(\boldsymbol{l}^T\mathbf{y}_1\right)+b\left(\boldsymbol{l}^T\mathbf{y}_2\right) \\ &= aT(\mathbf{y}_1)+bT(\mathbf{y}_2) \end{aligned} \qquad \textbf{(L8.3)}$$

Terlihat dari (L8.3) bahwa $T(\mathbf{y})$ merupakan penaksir yang linier terhadap kombinasi linier $\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$. Oleh karena $T(\mathbf{y})$ merupakan penaksir yang linier dan *unbiased* terhadap kombinasi linier $\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$, maka terbukti bahwa $\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$ dapat diestimasi oleh $T(\mathbf{y})$.

LAMPIRAN 9

Bukti (3.1.2).

$$E(\mathbf{y}\,|\,\mathbf{b}=\boldsymbol{\beta}) = E(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Z}\mathbf{b}+\mathbf{e}\,|\,\mathbf{b}=\boldsymbol{\beta})$$

$$= E(\mathbf{X}\boldsymbol{\alpha}\,|\,\mathbf{b}=\boldsymbol{\beta}) + E(\mathbf{Z}\mathbf{b}\,|\,\mathbf{b}=\boldsymbol{\beta}) + E(\mathbf{e}\,|\,\mathbf{b}=\boldsymbol{\beta})$$

$$= \mathbf{X}\boldsymbol{\alpha}+\mathbf{Z}\boldsymbol{\beta} \qquad \text{[karena } E(\mathbf{e})=\mathbf{0}\text{]}$$

$$Var(\mathbf{y}\,|\,\mathbf{b}=\boldsymbol{\beta}) = E\left((\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}-\mathbf{Z}\mathbf{b})(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}-\mathbf{Z}\mathbf{b})^T\,|\,\mathbf{b}=\boldsymbol{\beta}\right)$$

$$= E\left(\mathbf{e}\mathbf{e}^T\,|\,\mathbf{b}=\boldsymbol{\beta}\right)$$

$$= \mathbf{R}.$$

Oleh karena **b** dan **e** berdistribusi normal, maka **y|b** pun berdistribusi normal atau dari (2.4.b) ditulis sebagai $\mathbf{y}\,|\,\mathbf{b} \sim N(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Z}\boldsymbol{\beta},\mathbf{R})$ sehingga dari (2.4.a) *pdf*-nya adalah sebagai berikut:

$$g(\mathbf{y}\,|\,\boldsymbol{\beta}) = \frac{1}{(2\pi)^{n/2}|\mathbf{R}|^{1/2}}\exp\left(-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}-\mathbf{Z}\boldsymbol{\beta})^T\mathbf{R}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}-\mathbf{Z}\boldsymbol{\beta})\right)$$

$$= \frac{1}{(2\pi)^{n/2}|\mathbf{R}|^{1/2}}\exp\left(-\frac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{R}^{-1}\boldsymbol{\varepsilon}\right)$$

$$= g(\boldsymbol{\varepsilon}).$$

LAMPIRAN 10

Bukti (3.1.3).

$$
\begin{aligned}
\ln L(\boldsymbol{\alpha},\boldsymbol{\beta};\mathbf{y}) &= -\ln(2\pi)^n\left(|\mathbf{G}||\mathbf{R}|\right)^{1/2}-\frac{1}{2}\left(\boldsymbol{\varepsilon}^T\mathbf{R}^{-1}\boldsymbol{\varepsilon}+\boldsymbol{\beta}^T\mathbf{G}^{-1}\boldsymbol{\beta}\right)\\
&= -\ln(2\pi)^n\left(|\mathbf{G}||\mathbf{R}|\right)^{1/2}-\frac{1}{2}\left((\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}-\mathbf{Z}\boldsymbol{\beta})^T\mathbf{R}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}-\mathbf{Z}\boldsymbol{\beta})+\boldsymbol{\beta}^T\mathbf{G}^{-1}\boldsymbol{\beta}\right)\\
&= -\ln(2\pi)^n\left(|\mathbf{G}||\mathbf{R}|\right)^{1/2}-\frac{1}{2}\left(\left(\mathbf{y}^T-\boldsymbol{\alpha}^T\mathbf{X}^T-\boldsymbol{\beta}^T\mathbf{Z}^T\right)\mathbf{R}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}-\mathbf{Z}\boldsymbol{\beta})+\boldsymbol{\beta}^T\mathbf{G}^{-1}\boldsymbol{\beta}\right)
\end{aligned}
$$

$$
\begin{aligned}
=-\ln(2\pi)^n\left(|\mathbf{G}||\mathbf{R}|\right)^{1/2}-\frac{1}{2}(&\mathbf{y}^T\mathbf{R}^{-1}\mathbf{y}-\mathbf{y}^T\mathbf{R}^{-1}\mathbf{X}\boldsymbol{\alpha}-\mathbf{y}^T\mathbf{R}^{-1}\mathbf{Z}\boldsymbol{\beta}-\boldsymbol{\alpha}^T\mathbf{X}^T\mathbf{R}^{-1}\mathbf{y}+\boldsymbol{\alpha}^T\mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\boldsymbol{\alpha}+\boldsymbol{\alpha}^T\mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\boldsymbol{\beta}\\
&-\boldsymbol{\beta}^T\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{y}+\boldsymbol{\beta}^T\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\boldsymbol{\alpha}+\boldsymbol{\beta}^T\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\boldsymbol{\beta}+\boldsymbol{\beta}^T\mathbf{G}^{-1}\boldsymbol{\beta})
\end{aligned}
$$

LAMPIRAN 11

Bukti (3.1.4) dan (3.1.5).

Dengan menggunakan (2.1.3.a), (2.3.2.a), dan (2.3.2.b) didapat

- $\frac{\partial \ln L(\alpha, \beta; \mathbf{y})}{\partial \alpha} = \mathbf{0}$

$$\Leftrightarrow -\left(\mathbf{y}^T\mathbf{R}^{-1}\mathbf{X}\right)^T - \mathbf{X}^T\mathbf{R}^{-1}\mathbf{y} + \mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + \left(\tilde{\alpha}^T\mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\right)^T + \mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} + \left(\tilde{\beta}^T\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\right)^T = \mathbf{0}$$

$$\Leftrightarrow -\mathbf{X}^T\mathbf{R}^{-1}\mathbf{y} - \mathbf{X}^T\mathbf{R}^{-1}\mathbf{y} + \mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + \mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + \mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} + \mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} = \mathbf{0}$$

$$\Leftrightarrow -2\mathbf{X}^T\mathbf{R}^{-1}\mathbf{y} + 2\mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + 2\mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} = \mathbf{0}$$

$$\Leftrightarrow \mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + \mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} = \mathbf{X}^T\mathbf{R}^{-1}\mathbf{y}$$

- $\frac{\partial \ln L(\alpha, \beta; \mathbf{y})}{\partial \beta} = \mathbf{0}$

$$\Leftrightarrow -\left(\mathbf{y}^T\mathbf{R}^{-1}\mathbf{Z}\right)^T + \left(\tilde{\alpha}^T\mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\right)^T - \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{y} + \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} + \left(\tilde{\beta}^T\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\right)^T + \mathbf{G}^{-1}\tilde{\beta} + \left(\tilde{\beta}^T\mathbf{G}^{-1}\right)^T = \mathbf{0}$$

$$\Leftrightarrow -\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{y} + \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} - \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{y} + \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} + \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} + \mathbf{G}^{-1}\tilde{\beta} + \mathbf{G}^{-1}\tilde{\beta} = \mathbf{0}$$

$$\Leftrightarrow -2\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{y} + 2\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + 2\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\tilde{\beta} + 2\mathbf{G}^{-1}\tilde{\beta} = \mathbf{0}$$

$$\Leftrightarrow \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha} + \left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)\tilde{\beta} = \mathbf{Z}^T\mathbf{R}^{-1}\mathbf{y}$$

LAMPIRAN 12

Bukti (3.1.6) dan (3.1.7).

Taksiran parameter $\tilde{\alpha}$ dan $\tilde{\beta}$ dengan asumsi **b** dan **e** berdistribusi normal didapat dengan mengeliminasi $\tilde{\beta}$ dari (3.1.4) dan (3.1.5), yaitu mengalikan (3.1.4) dengan $\mathbf{I}$ dan mengalikan $\mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}$ dengan (3.1.5), kemudian kedua perkalian tersebut dikurangkan. Hasil eliminasinya adalah sebagai berikut:

$$\mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha}-\mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{X}\tilde{\alpha}=\mathbf{X}^T\mathbf{R}^{-1}\mathbf{y}-\mathbf{X}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{y}$$

$$\Leftrightarrow \mathbf{X}^T\left(\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\right)\mathbf{X}\tilde{\alpha}=\mathbf{X}^T\left(\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\right)\mathbf{y}$$

sehingga

$$\tilde{\alpha}=\left(\mathbf{X}^T\left(\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\right)\mathbf{X}\right)^{-1}\mathbf{X}^T\left(\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\right)\mathbf{y}$$

sedangkan

$$\tilde{\beta}=\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\left(\mathbf{y}-\mathbf{X}\tilde{\alpha}\right).$$

LAMPIRAN 13

Bukti bahwa $\mathbf{W} = \mathbf{\Omega}^{-1}$.

$$
\begin{aligned}
\mathbf{\Omega W} &= \left(\mathbf{R} + \mathbf{ZGZ}^T\right)\left(\mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\right) \\
&= \mathbf{RR}^{-1} - \mathbf{RR}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1} + \mathbf{ZGZ}^T\mathbf{R}^{-1} - \mathbf{ZGZ}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1} \\
&= \mathbf{I} - \mathbf{IZ}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1} + \mathbf{ZGZ}^T\mathbf{R}^{-1} - \mathbf{ZGZ}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1} \\
&= \mathbf{I} + \mathbf{ZGZ}^T\mathbf{R}^{-1} - \mathbf{Z}\left(\mathbf{GZ}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{I}\right)\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1} \\
&= \mathbf{I} + \mathbf{ZGZ}^T\mathbf{R}^{-1} - \mathbf{ZG}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1} \\
&= \mathbf{I} + \mathbf{ZGZ}^T\mathbf{R}^{-1} - \mathbf{ZGIZ}^T\mathbf{R}^{-1} \\
&= \mathbf{I} + \mathbf{ZGZ}^T\mathbf{R}^{-1} - \mathbf{ZGZ}^T\mathbf{R}^{-1} \\
&= \mathbf{I}
\end{aligned}
$$

LAMPIRAN 14

Bukti bahwa $\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}=\mathbf{G}\mathbf{Z}^T\mathbf{\Omega}^{-1}$.

$$\begin{aligned}
\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1} &= \left(\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}+\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}-\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\right)\mathbf{Z}^T\mathbf{R}^{-1}\\
&= \left(\left(\mathbf{I}+\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\right)\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}-\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\right)\mathbf{Z}^T\mathbf{R}^{-1}\\
&= \left(\left(\mathbf{G}\mathbf{G}^{-1}+\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\right)\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}-\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\right)\mathbf{Z}^T\mathbf{R}^{-1}\\
&= \left(\mathbf{G}\left(\mathbf{G}^{-1}+\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\right)\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}-\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\right)\mathbf{Z}^T\mathbf{R}^{-1}\\
&= \left(\mathbf{G}-\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\right)\mathbf{Z}^T\mathbf{R}^{-1}\\
&= \left(\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}-\mathbf{G}\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\right)\\
&= \mathbf{G}\mathbf{Z}^T\left(\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^T\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)^{-1}\mathbf{Z}^T\mathbf{R}^{-1}\right)\\
&= \mathbf{G}\mathbf{Z}^T\mathbf{\Omega}^{-1} \qquad \text{[karena (3.2.1)]}
\end{aligned}$$

LAMPIRAN 15

Bukti (3.3.1).

$$
\begin{aligned}
E(\mathbf{y}) &= E(\mathbf{X}\boldsymbol{\alpha} + \mathbf{Zb} + \mathbf{e}) \\
&= E(\mathbf{X}\boldsymbol{\alpha}) + E(\mathbf{Zb}) + E(\mathbf{e}) \\
&= \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}E(\mathbf{b}) + E(\mathbf{e}) \\
&= \mathbf{X}\boldsymbol{\alpha} \qquad \text{[karena } E(\mathbf{b}) = \mathbf{0} \text{ dan } E(\mathbf{e}) = \mathbf{0}\text{]}
\end{aligned}
$$

$$
\begin{aligned}
Var(\mathbf{y}) &= E\left((\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})^T\right) \\
&= E\left((\mathbf{Zb} + \mathbf{e})(\mathbf{Zb} + \mathbf{e})^T\right) \\
&= E\left((\mathbf{Zb} + \mathbf{e})(\mathbf{b}^T\mathbf{Z}^T + \mathbf{e}^T)\right) \\
&= E\left(\mathbf{Zbb}^T\mathbf{Z}^T + \mathbf{Zbe}^T + \mathbf{eb}^T\mathbf{Z}^T + \mathbf{ee}^T\right) \\
&= E\left(\mathbf{Zbb}^T\mathbf{Z}^T\right) + E\left(\mathbf{Zbe}^T\right) + E\left(\mathbf{eb}^T\mathbf{Z}^T\right) + E\left(\mathbf{ee}^T\right) \\
&= \mathbf{Z}E\left(\mathbf{bb}^T\right)\mathbf{Z}^T + \mathbf{Z}E\left(\mathbf{be}^T\right) + E\left(\mathbf{eb}^T\right)\mathbf{Z}^T + \mathbf{R} \\
&= \mathbf{ZGZ}^T + \mathbf{R} \qquad \text{[karena } E\left(\mathbf{be}^T\right) = E\left(\mathbf{eb}^T\right) = \mathbf{0}\text{]} \\
&= \boldsymbol{\Omega} \qquad \text{[karena } \boldsymbol{\Omega} = \mathbf{R} + \mathbf{ZGZ}^T.\text{]}
\end{aligned}
$$

Oleh karena **b** dan **e** berdistribusi normal, maka **y** pun berdistribusi normal atau ditulis sebagai $\mathbf{y} \sim N(\mathbf{X}\boldsymbol{\alpha}, \boldsymbol{\Omega})$ di mana pdf-nya adalah sebagai berikut:

$$
f(\mathbf{y}) = \frac{1}{(2\pi)^{n/2} \left|\boldsymbol{\Omega}(\delta)\right|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})^T \boldsymbol{\Omega}^{-1}(\delta)(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})\right)
$$

LAMPIRAN 16

Bukti (3.3.3).

$$s_j(\boldsymbol{\alpha},\boldsymbol{\delta}) = \frac{\partial \ln L(\boldsymbol{\alpha},\boldsymbol{\delta};\mathbf{y})}{\partial \delta_j} = \frac{\partial\left(c - \frac{1}{2}\left(\ln\left(\left|\boldsymbol{\Omega}(\boldsymbol{\delta})\right|\right) + (\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T \boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)\right)}{\partial \delta_j}$$

$$= \frac{\partial c}{\partial \delta_j} - \frac{1}{2}\frac{\partial \ln\left(\left|\boldsymbol{\Omega}(\boldsymbol{\delta})\right|\right)}{\partial \delta_j} - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T \frac{\partial \boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})}{\partial \delta_j}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})$$

$$= -\frac{1}{2} tr\left(\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\frac{\partial \boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial \delta_j}\right) + \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\left(\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\frac{\partial \boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial \delta_j}\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})$$

LAMPIRAN 17

Bukti (3.3.5).

Akan dicari $\Upsilon(\delta)$ sehingga *Scoring Algorithm* dapat digunakan. Misalkan $\Upsilon_{jk}(\delta)$ adalah elemen ke-*jk* dari $\Upsilon(\delta)$, yaitu:

$$\Upsilon_{jk}(\boldsymbol{\delta}) = -E\left(\frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\delta};\mathbf{y})}{\partial\delta_j \partial\delta_k}\right)$$

sedangkan $\boldsymbol{\Omega}_{(j)} = \frac{\partial\boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial\delta_j}$ dan $\boldsymbol{\Omega}_{(j,k)} = \frac{\partial\boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial\delta_k\partial\delta_j}$.

$$\Upsilon_{jk}(\boldsymbol{\delta}) = -E\left(\frac{\partial^2\left(c - \frac{1}{2}\left(\ln\left(|\boldsymbol{\Omega}|\right) + (\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T \boldsymbol{\Omega}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)\right)}{\partial\delta_k\partial\delta_j}\right)$$

$$= -E\left(\frac{\partial^2 c}{\partial\delta_k\partial\delta_j} - \frac{1}{2}\frac{\partial^2\ln\left(|\boldsymbol{\Omega}|\right)}{\partial\delta_k\partial\delta_j} - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\frac{\partial^2\boldsymbol{\Omega}^{-1}}{\partial\delta_k\partial\delta_j}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)$$

$$= -E\left(\begin{array}{l} -\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\frac{\partial^2\boldsymbol{\Omega}}{\partial\delta_k\partial\delta_j}\right) - tr\left(\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_k}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_j}\right)\right) - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T \\ \left(-\boldsymbol{\Omega}^{-1}\frac{\partial^2\boldsymbol{\Omega}}{\partial\delta_k\partial\delta_j}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_k}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_j}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_j}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_k}\boldsymbol{\Omega}^{-1}\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}) \end{array}\right)$$

$$= -E\left(\begin{array}{l} -\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\right) - tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\right)\right) - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T \\ \left(-\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}) \end{array}\right)$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\right) - tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\right)\right) - \frac{1}{2}tr\left((\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\left(\begin{array}{l} -\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1} \end{array}\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)\right)$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\right) - tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\right)\right) - \frac{1}{2}tr\left(\begin{pmatrix} -\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\mathbf{\Omega}^{-1} \\ +\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1} \\ +\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1} \end{pmatrix}(\mathbf{y}-\mathbf{X\alpha})(\mathbf{y}-\mathbf{X\alpha})^T\right)\right)$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\right) - tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\right) + tr\left(\begin{pmatrix} -\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\mathbf{\Omega}^{-1} \\ +\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1} \\ +\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1} \end{pmatrix}(\mathbf{y}-\mathbf{X\alpha})(\mathbf{y}-\mathbf{X\alpha})^T\right)\right)\right)$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)} - \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)} + \begin{pmatrix} -\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\mathbf{\Omega}^{-1} \\ +\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1} \\ +\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1} \end{pmatrix}(\mathbf{y}-\mathbf{X\alpha})(\mathbf{y}-\mathbf{X\alpha})^T\right)\right)\right)$$

$$= \frac{1}{2}tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)} - \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)} + \begin{pmatrix} -\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\mathbf{\Omega}^{-1} + \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1} \\ +\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1} \end{pmatrix} E\left((\mathbf{y}-\mathbf{X\alpha})(\mathbf{y}-\mathbf{X\alpha})^T\right)\right)$$

$$= \frac{1}{2}tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)} - \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)} + \left(-\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\mathbf{\Omega}^{-1} + \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1} + \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\right)\mathbf{\Omega}\right)$$

$$= \frac{1}{2}tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)} - \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)} + \left(-\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)}\mathbf{\Omega}^{-1}\mathbf{\Omega} + \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega} + \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}\right)\right)$$

$$= \frac{1}{2}tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)} - \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)} - \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j,k)} + \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)} + \mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\right)$$

$$= \frac{1}{2}tr\left(\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(j)}\mathbf{\Omega}^{-1}\mathbf{\Omega}_{(k)}\right).$$

LAMPIRAN 18

Bukti bahwa $\hat{\tau}\left(\hat{\boldsymbol{\delta}}(\mathbf{y}),\mathbf{y}\right)$ Tetap *Unbiased* untuk Mengestimasi $\tau = \boldsymbol{\lambda}^T\boldsymbol{\alpha} + \boldsymbol{\omega}^T\boldsymbol{\beta}$.

Dengan menggunakan (2.7.3.c) dan (2.7.3.d) maka didapatkan

$$
\begin{aligned}
\boldsymbol{\lambda}^T\hat{\boldsymbol{\alpha}}(\delta,\mathbf{y}) - \boldsymbol{\lambda}^T\boldsymbol{\alpha} &= \boldsymbol{\lambda}^T\left(\hat{\boldsymbol{\alpha}}(\delta,\mathbf{y}) - \boldsymbol{\alpha}\right) \\
&= \boldsymbol{\lambda}^T\left(\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{y} - \left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\boldsymbol{\alpha}\right) \\
&= \boldsymbol{\lambda}^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha}) \\
&= \boldsymbol{\lambda}^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)(\mathbf{Zb} + \mathbf{e}) \qquad \textbf{(L18.1)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\boldsymbol{\beta}}(\delta,\mathbf{y}) &= \mathbf{G}(\delta)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}(\delta)\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\alpha}}(\delta,\mathbf{y})\right) \\
&= \mathbf{G}(\delta)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}(\delta)\left(\mathbf{y} - \mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{y}\right) \\
&= \mathbf{G}(\delta)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}(\delta)\left(\mathbf{I} - \mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\right)\mathbf{y} \\
&= \mathbf{G}(\delta)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}(\delta)\left(\mathbf{I} - \mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\right)(\mathbf{X}\boldsymbol{\alpha} + \mathbf{Zb} + \mathbf{e}) \\
&= \mathbf{G}(\delta)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}(\delta)\begin{pmatrix} \mathbf{X}\boldsymbol{\alpha} - \mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\boldsymbol{\alpha} + \mathbf{Zb} + \mathbf{e} \\ -\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)(\mathbf{Zb} + \mathbf{e}) \end{pmatrix} \\
&= \mathbf{G}(\delta)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}(\delta)\left(\mathbf{X}\boldsymbol{\alpha} - \mathbf{X}\boldsymbol{\alpha} + \left(\mathbf{I} - \mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\right)(\mathbf{Zb} + \mathbf{e})\right) \\
&= \mathbf{G}(\delta)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}(\delta)\left(\mathbf{I} - \mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\right)(\mathbf{Zb} + \mathbf{e}). \qquad \textbf{(L18.2)}
\end{aligned}
$$

Dengan memisalkan $\mathbf{P}(\delta) = \mathbf{I} - \mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\delta)$ maka persamaan (L18.2) dapat ditulis menjadi

$$\hat{\beta}(\delta,\mathbf{y}) = \mathbf{G}(\delta)\mathbf{Z}^T\Omega^{-1}(\delta)\mathbf{P}(\delta)(\mathbf{Zb}+\mathbf{e}). \qquad \textbf{(L18.3)}$$

Berdasarkan (L18.1) dan (L18.3) maka

$$\begin{aligned}\hat{\tau}\left(\hat{\delta}(\mathbf{y}),\mathbf{y}\right) - \mathrm{t} &= \lambda^T\hat{\alpha}\left(\hat{\delta}(\mathbf{y}),\mathbf{y}\right) + \omega^T\hat{\beta}\left(\hat{\delta}(\mathbf{y}),\mathbf{y}\right) - \lambda^T\alpha - \omega^T\mathbf{b}\\ &= \lambda^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\left(\hat{\delta}(\mathbf{y})\right)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\left(\hat{\delta}(\mathbf{y})\right)(\mathbf{Zb}+\mathbf{e})\\ &\quad +\omega^T\mathbf{G}\left(\hat{\delta}(\mathbf{y})\right)\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\left(\hat{\delta}(\mathbf{y})\right)\mathbf{P}\left(\hat{\delta}(\mathbf{y})\right)(\mathbf{Zb}+\mathbf{e}) - \omega^T\mathbf{b}.\end{aligned}$$

Dengan menggunakan (3.4.2) dan memisalkan

$$\begin{aligned}\phi(\mathbf{b},\mathbf{e}) &= \lambda^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{X}\right)^{-1}\mathbf{X}^T\Omega^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)(\mathbf{Zb}+\mathbf{e})\\ &\quad +\omega^T\mathbf{G}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{Z}^T\Omega^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{P}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)(\mathbf{Zb}+\mathbf{e})\end{aligned}$$

maka $\hat{\tau}\left(\hat{\delta}(\mathbf{y}),\mathbf{y}\right) - \mathrm{t} = \phi(\mathbf{b},\mathbf{e}) - \omega^T\mathbf{b}$. Berdasarkan (3.4.2) juga didapat

$$\begin{aligned}\phi(-\mathbf{b},-\mathbf{e}) &= \lambda^T\left(\mathbf{X}^T\Omega^{-1}\left(\hat{\delta}(-(\mathbf{Zb}+\mathbf{e}))\right)\mathbf{X}\right)^{-1}\mathbf{X}^T\Omega^{-1}\left(\hat{\delta}(-(\mathbf{Zb}+\mathbf{e}))\right)(-(\mathbf{Zb}+\mathbf{e}))\\ &\quad +\omega^T\mathbf{G}\left(\hat{\delta}(-(\mathbf{Zb}+\mathbf{e}))\right)\mathbf{Z}^T\Omega^{-1}\left(\hat{\delta}(-(\mathbf{Zb}+\mathbf{e}))\right)\mathbf{P}\left(\hat{\delta}(-(\mathbf{Zb}+\mathbf{e}))\right)(-(\mathbf{Zb}+\mathbf{e}))\\ &= -\lambda^T\left(\mathbf{X}^T\Omega^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{X}\right)^{-1}\mathbf{X}^T\Omega^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)(\mathbf{Zb}+\mathbf{e})\\ &\quad -\omega^T\mathbf{G}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{Z}^T\Omega^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{P}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)(\mathbf{Zb}+\mathbf{e})\\ &= -\begin{pmatrix}\lambda^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)(\mathbf{Zb}+\mathbf{e})\\ +\omega^T\mathbf{G}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{Z}^T\Omega^{-1}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)\mathbf{P}\left(\hat{\delta}(\mathbf{Zb}+\mathbf{e})\right)(\mathbf{Zb}+\mathbf{e})\end{pmatrix}\\ &= -\phi(\mathbf{b},\mathbf{e})\end{aligned}$$

sehingga dari (2.8.1.b) diketahui bahwa $\phi(\mathbf{b},\mathbf{e})$ merupakan fungsi ganjil dari $\mathbf{b}$ dan $\mathbf{e}$. Dengan asumsi bahwa $\mathbf{b}$ dan $\mathbf{e}$ berdistribusi normal (simetris) dengan *mean* nol, maka berdasarkan lemma 3.1 $\phi(\mathbf{b},\mathbf{e})$ berdistribusi simetris di sekitar nol. Dengan demikian $E\left(\phi(\mathbf{b},\mathbf{e})\right)=0$ dan

$$
\begin{aligned}
E\left(\hat{\tau}\left(\hat{\delta}(\mathbf{y}),\mathbf{y}\right)-\mathrm{t}\right) &= E\left(\phi(\mathbf{b},\mathbf{e})-\boldsymbol{\omega}^T\mathbf{b}\right) \\
&= E\left(\phi(\mathbf{b},\mathbf{e})\right)-E\left(\boldsymbol{\omega}^T\mathbf{b}\right) \\
&= 0-\boldsymbol{\omega}^T E(\mathbf{b}) \\
&= 0.
\end{aligned}
$$

LAMPIRAN 19

Data Ilustrasi Penggunaan Metode EBLUP pada *General Linear Mixed Model* serta Bentuk Matriks $\mathbf{X}$ dan $\mathbf{Z}$.

No.	*Location*	*Champaign*	*Sales*($)
1	1	3	267.80
2	1	2	248.84
3	1	2	247.89
4	1	2	251.21
5	1	1	276.48
6	2	1	229.77
7	2	1	231.92
8	2	1	232.52
9	2	3	219.36
10	2	3	212.42
11	3	3	181.26
12	3	1	201.34
13	3	2	143.37
14	3	2	144.46
15	3	3	186.87

No.	*Location*	*Champaign*	*Sales*($)
16	4	1	225.26
17	4	1	218.27
18	4	2	181.10
19	4	3	228.91
20	4	3	208.66
21	5	2	198.30
22	5	3	232.66
23	5	2	207.18
24	5	1	237.22
25	5	1	245.47
26	6	2	178.58
27	6	2	170.05
28	6	1	208.67
29	6	2	169.16
30	6	3	185.52

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

LAMPIRAN 20

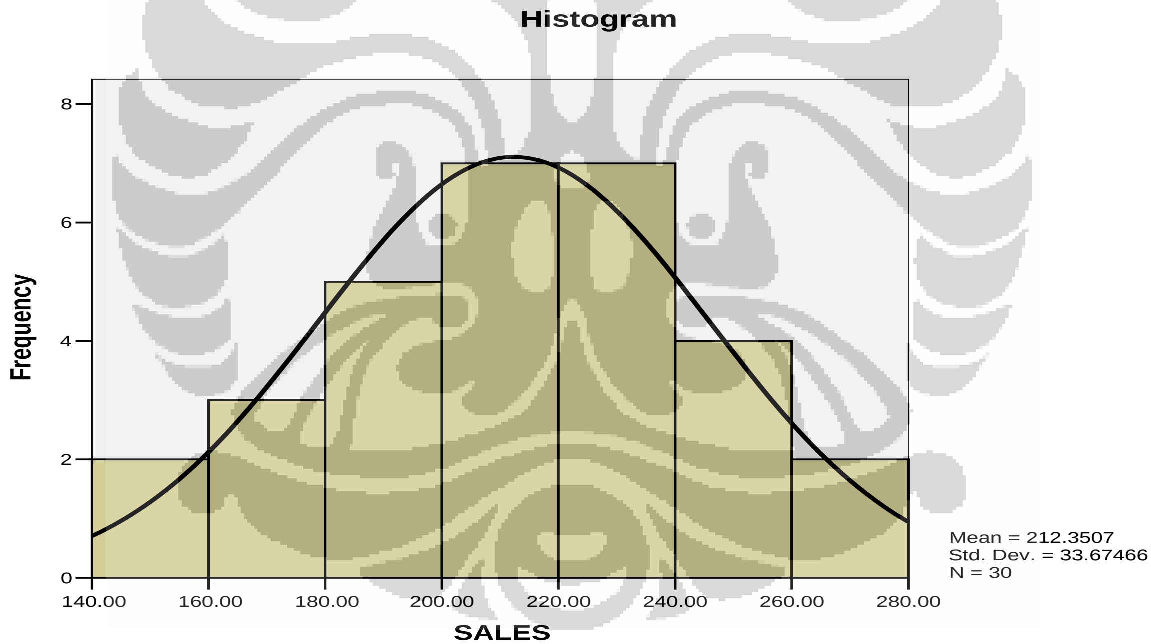
Pengujian Kenormalan.

Tests of Normality

	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
SALES	.089	30	.200*	.978	30	.778

*. This is a lower bound of the true significance.

a. Lilliefors Significance Correction



LAMPIRAN 21

Penaksiran Parameter pada *General Linear Mixed Model* Menggunakan

Software MATLAB 7.0.1.

Editor - D:\papa\Tugas Akhir\TA\apaan.m

File Edit Text Desktop Window Help

```
clear;
tic;
syms d0 d1;
X = [0 0 1; 0 1 0; 0 1 0; 0 1 0; 1 0 0;...
     1 0 0; 1 0 0; 1 0 0; 0 0 1; 0 0 1;...
     0 0 1; 1 0 0; 0 1 0; 0 1 0; 0 0 1;...
     1 0 0; 1 0 0; 0 1 0; 0 0 1; 0 0 1;...
     0 1 0; 0 0 1; 0 1 0; 1 0 0; 1 0 0;...
     0 1 0; 0 1 0; 1 0 0; 0 1 0; 0 0 1];
Y = [267.80; 248.84; 247.89; 251.21; 276.48;...
     229.77; 231.92; 232.52; 219.36; 212.42;...
     181.26; 201.34; 143.37; 144.46; 186.87;...
     225.26; 218.27; 181.10; 228.91; 208.66;...
     198.30; 232.66; 207.18; 237.22; 245.47;...
     178.58; 170.05; 208.67; 169.16; 185.52];
Z = [1 0 0 0 0 0; 1 0 0 0 0 0; 1 0 0 0 0 0; 1 0 0 0 0 0; 1 0 0 0 0 0;...
     0 1 0 0 0 0; 0 1 0 0 0 0; 0 1 0 0 0 0; 0 1 0 0 0 0; 0 1 0 0 0 0;...
     0 0 1 0 0 0; 0 0 1 0 0 0; 0 0 1 0 0 0; 0 0 1 0 0 0; 0 0 1 0 0 0;...
     0 0 0 1 0 0; 0 0 0 1 0 0; 0 0 0 1 0 0; 0 0 0 1 0 0; 0 0 0 1 0 0;...
     0 0 0 0 1 0; 0 0 0 0 1 0; 0 0 0 0 1 0; 0 0 0 0 1 0; 0 0 0 0 1 0;...
     0 0 0 0 0 1; 0 0 0 0 0 1; 0 0 0 0 0 1; 0 0 0 0 0 1; 0 0 0 0 0 1];
delta = [0.01; 0.01];
epsilon = [0.01; 0.01];
err = [1; 1];
```

```
R = d0*eye(30);
G = d1*eye(6);
O = R + Z*G*Z';
a_cap = inv(X'*inv(O)*X)*(X'*inv(O)*Y);
diffy(1,:,:) = diff(O, d0);
diffy(2,:,:) = diff(O, d1);

for j=1:2
    p = diffy(j,:,:);
    p = reshape(p, 30, 30);
    S(j) = -1/2*trace(inv(O)*p) + 1/2*(Y - X*a_cap)'*...
          (inv(O)*p*inv(O))*(Y-X*a_cap);
    for k=1:2
        q = diffy(k,:,:);
        q = reshape(q, 30, 30);
        L(j,k) = 1/2*trace((inv(O)*p) * (inv(O)*q));
    end
end

while norm(err,inf) >= epsilon
    nS = subs(S, {d0, d1}, {delta(1), delta(2)});
    nL = subs(L, {d0, d1}, {delta(1), delta(2)});

    del = delta;
    delta = del + inv(nL)*nS';

    err = abs(delta - del);
 end

 R = delta(1)*eye(30);
 G = delta(2)*eye(6);
 O = R + Z*G*Z';
 a_cap = inv(X'*inv(O)*X)*(X'*inv(O)*Y)
 b_cap = G*Z'*inv(O)*(Y - X*a_cap)
 y_cap = X*a_cap + Z*b_cap
 Y - y_cap
 t = toc;
 fprintf('Waktu: %f', t);
```

taksiranmama.m | MAMA.m | coba.m | apaan.m

script | Ln 1 | Col 1 | OVR

LAMPIRAN 22

Tabel Perbandingan antara $\mathbf{y}$ dan $\hat{\mathbf{y}}$.

Sales dari Data	*Sales* dari Taksiran
267.80	271.979
248.84	244.5921
247.89	244.5921
251.21	244.5921
276.48	283.512
229.77	229.8142
231.92	229.8142
232.52	229.8142
219.36	218.2811
212.42	218.2811
181.26	180.5283
201.34	192.0614
143.37	153.1415
144.46	153.1415
186.87	180.5283
225.26	224.8945
218.27	224.8945
181.10	185.9746
228.91	213.3615

208.66	213.3615
198.30	202.9899
232.66	230.3768
207.18	202.9899
237.22	241.9098
245.47	241.9098
178.58	169.3754
170.05	169.3754
208.67	208.2953
169.16	169.3754
185.52	196.7623

LAMPIRAN 23

Tabel *Error*.

Error
-4.179
4.2479
3.2979
6.6179
-7.032
-0.0442
2.1058
2.7058
1.0789
-5.8611
0.7317
9.2786
-9.7715
-8.6815
6.3417

Error
0.3655
-6.6245
-4.8746
15.5485
-4.7015
-4.6899
2.2832
4.1901
-4.6898
3.5602
9.2046
0.6746
0.3747
-0.2154
-11.2423