LAMPIRAN

LAMPIRAN 1

Menunjukkan $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta}) \leftrightarrow \boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$

Keterangan: $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$.

Bukti:

$(\rightarrow)$ Adit: $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta}) \rightarrow \boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$

Oleh karena $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta})$, maka:

- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$
- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$

  $\vdots$
- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_p} = 0,$
- $\dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \ i = 1, 2, \ldots, p,$
- $D = \dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} . \dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0, \ i = 1, 2, \ldots, p, j = 1, 2, \ldots, p \ , i \neq j$.

Akan ditunjukkan bahwa $\boldsymbol{\theta}$ juga memaksimumkan $\ln L(\boldsymbol{\theta})$, yaitu:

- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = \frac{1}{L(\boldsymbol{\theta})} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = \frac{1}{L(\boldsymbol{\theta})} .0 = 0$$

$$\therefore \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$$

- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = 0$

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = \frac{1}{L(\boldsymbol{\theta})} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = \frac{1}{L(\boldsymbol{\theta})} .0 = 0$$

$$\therefore \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$$

$$\vdots$$

- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_p} = 0$

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_p} = \frac{1}{L(\boldsymbol{\theta})} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_p} = \frac{1}{L(\boldsymbol{\theta})} .0 = 0$$

$$\therefore \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_p} = 0,$$

- $\dfrac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0,\ i = 1,2,\ldots,p,$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2} = \frac{\partial}{\partial\theta_i}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_i}\right) = \frac{\partial}{\partial\theta_i}\left(\frac{1}{L(\boldsymbol{\theta})}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\right)$$

$$= \frac{\partial}{\partial\theta_i}\left(\frac{1}{L(\boldsymbol{\theta})}\right)\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i} + \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{1}{L(\boldsymbol{\theta})}$$

$$= \frac{\partial}{\partial\theta_i}\left(\frac{1}{L(\boldsymbol{\theta})}\right)\cdot 0 + \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{1}{L(\boldsymbol{\theta})}$$

$$= \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{1}{L(\boldsymbol{\theta})} < 0\cdot\frac{1}{L(\boldsymbol{\theta})} = 0$$

$$\therefore \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2} < 0,\ i = 1,2,\ldots,p,$$

- $$D = \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2}.\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2 > 0,\ i = 1,2,\ldots,p, j = 1,2,\ldots,p\ , i \neq j$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2} = \frac{\partial}{\partial\theta_i}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_i}\right) = \frac{\partial}{\partial\theta_i}\left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L(\boldsymbol{\theta})}\right), i = 1,2,\ldots,p$$

$$= \frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L^2(\boldsymbol{\theta})}$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2} = \frac{\partial}{\partial\theta_j}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_j}\right) = \frac{\partial}{\partial\theta_j}\left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L(\boldsymbol{\theta})}\right), j = 1,2,\ldots,p$$

$$= \frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L^2(\boldsymbol{\theta})}$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left(\frac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_j}\right) = \frac{\partial}{\partial\theta_i}\left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L(\boldsymbol{\theta})}\right), i = 1,2,\ldots,p, j = 1,2,\ldots,p$$

$$= \frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L^2(\boldsymbol{\theta})}.$$

$$D = \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2$$

$$= \left(\frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L^2(\boldsymbol{\theta})}\right)\cdot\left(\frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}}{L^2(\boldsymbol{\theta})}\right)$$

$$-\left(\frac{\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot L(\boldsymbol{\theta}) - \frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}}{L^2(\boldsymbol{\theta})}\right)^2, i = 1,2,\cdots,p,\ j = 1,2,\cdots,p$$

$$= \frac{1}{L^4(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot L^2(\boldsymbol{\theta}) + \left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\right.$$

$$\left. - \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\cdot L(\boldsymbol{\theta}) - \frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\right)^2\cdot L(\boldsymbol{\theta})\right]$$

$$- \frac{1}{L^4(\boldsymbol{\theta})}\left[\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\cdot L^2(\boldsymbol{\theta}) + \left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\right.$$

$$\left. -2\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot L(\boldsymbol{\theta})\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)\right]$$

$$= \frac{L^2(\boldsymbol{\theta})}{L^4(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\right]$$

$$+\frac{L(\boldsymbol{\theta})}{L^4(\boldsymbol{\theta})}\left[2\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_j}\right)^2\right.$$

$$\left.-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot\left(\frac{\partial L(\boldsymbol{\theta})}{\partial\theta_i}\right)^2\right]$$

$$= \frac{1}{L^2(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\right]$$

$$+\frac{1}{L^3(\boldsymbol{\theta})}\left[2\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\cdot 0\cdot 0-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot(0)^2-\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}\cdot(0)^2\right]$$

$$= \frac{1}{L^2(\boldsymbol{\theta})}\left[\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i^2}\cdot\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2\right] > \frac{1}{L^2(\boldsymbol{\theta})}\cdot 0 = 0$$

$$\therefore D = \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i^2}.\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_j^2}-\left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2 > 0,\ i=1,2,\ldots,p, j=1,2,\ldots,p\ ,i\neq j.$$

Dengan perkataan lain terbukti bahwa $\boldsymbol{\theta}$ juga memaksimumkan $\ln L(\boldsymbol{\theta})$.

$(\leftarrow)$ Adit: $\boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$ $\rightarrow$ $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta})$

Oleh karena $\boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$, maka

- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_1}=0,$
- $\dfrac{\partial \ln L(\boldsymbol{\theta})}{\partial\theta_2}=0,$

$\vdots$

- $$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_n} = 0,$$

- $$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0,\ i = 1,2,\ldots,p,$$

- $$D = \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} . \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0,\ i = 1,2,\ldots,p,\ j = 1,2,\ldots,p\ ,i \neq j.$$

Akan ditunjukkan bahwa $\boldsymbol{\theta}$ juga memaksimumkan $L(\boldsymbol{\theta})$, yaitu:

- $$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$$

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = L(\boldsymbol{\theta}) \cdot \frac{1}{L(\boldsymbol{\theta})} . \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_1} = L(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_1} = 0,$$

- $$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$$

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = L(\boldsymbol{\theta}) \cdot \frac{1}{L(\boldsymbol{\theta})} . \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_2} = L(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_2} = 0,$$

$$\vdots$$

- $\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_p} = 0,$

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_p} = L(\boldsymbol{\theta}) \cdot \frac{1}{L(\boldsymbol{\theta})} . \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_p} = L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_p} = L(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_p} = 0,$$

- $\dfrac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \; i = 1, 2, \ldots, p,$

$$\begin{aligned}
\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} &= \frac{\partial \left(\dfrac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)}{\partial \theta_i} \\
&= \frac{\partial}{\partial \theta_i} \left(L(\boldsymbol{\theta}) \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_i} \right) \\
&= \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_i} + L(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \\
&= \frac{1}{L(\boldsymbol{\theta})} \cdot \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \\
&= \frac{1}{L(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \right] \\
&= \frac{1}{L(\boldsymbol{\theta})} \cdot \left[(0)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \right] \\
&= L(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} < L(\boldsymbol{\theta}) \cdot 0 = 0
\end{aligned}$$

$$\therefore \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} < 0, \; i = 1, 2, \ldots, p,$$

- $D = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} . \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0,\ i = 1,2,\ldots,p, j = 1,2,\ldots,p\ , i \neq j$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} = \frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2}}{L(\boldsymbol{\theta})}, i = 1,2,\ldots,p$$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} = \frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2}}{L(\boldsymbol{\theta})}, j = 1,2,\ldots,p$$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} = \frac{\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}}{L(\boldsymbol{\theta})}, i = 1,2,\ldots,p, j = 1,2,\ldots,p$$

$$D = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2$$

$$= \left(\frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2}}{L(\boldsymbol{\theta})} \right) \cdot \left(\frac{\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2}}{L(\boldsymbol{\theta})} \right)$$

$$- \left(\frac{\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} + L^2(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}}{L(\boldsymbol{\theta})} \right)^2, i = 1,2,\ldots,p, j = 1,2,\ldots,p, i \neq j$$

$$= \frac{1}{L^2(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^4(\boldsymbol{\theta}) \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + \right.$$

$$\left. + L^2(\boldsymbol{\theta}) \cdot \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + L^2(\boldsymbol{\theta}) \cdot \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \right]$$

$$- \frac{1}{L^2(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 + L^4(\boldsymbol{\theta}) \cdot \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 + 2 \cdot L^2(\boldsymbol{\theta}) \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right]$$

$$= \frac{L^4(\boldsymbol{\theta})}{L^2(\boldsymbol{\theta})} \cdot \left[\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 \right]$$

$$+ \frac{L^2(\boldsymbol{\theta})}{L^2(\boldsymbol{\theta})} \cdot \left[\left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + \left(\frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \right)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} - 2 \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial L(\boldsymbol{\theta})}{\partial \theta_j} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right]$$

$$= L^2(\boldsymbol{\theta}) \cdot \left[\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 \right]$$

$$+ 1 \cdot \left[(0)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} + (0)^2 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} - 2 \cdot 0 \cdot 0 \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right]$$

$$= L^2(\boldsymbol{\theta}) \cdot \left[\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i^2} \cdot \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 \right] > L^2(\boldsymbol{\theta}) \cdot 0 = 0$$

$$\therefore D = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i^2} . \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_j^2} - \left(\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)^2 > 0,\ i = 1,2,\ldots,p, j = 1,2,\ldots,p\ , i \neq j$$

Dengan perkataan lain, terbukti bahwa $\boldsymbol{\theta}$ juga memaksimumkan $L(\boldsymbol{\theta})$.

Terbukti bahwa $\boldsymbol{\theta}$ memaksimumkan $L(\boldsymbol{\theta}) \leftrightarrow \boldsymbol{\theta}$ memaksimumkan $\ln L(\boldsymbol{\theta})$,

$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$.

LAMPIRAN 2

Mencari matriks **T**, dimana $\mathbf{T}\,\mathbf{T}' = \mathbf{\Sigma}_\delta$, dengan dekomposisi Cholesky.

Misalkan $\mathbf{T} = \begin{pmatrix} t_{00} & 0 & \cdots & 0 \\ t_{10} & t_{11} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ t_{R0} & t_{R1} & \cdots & t_{RR} \end{pmatrix}$

- $t_{00} = \sigma_{\delta_1}$
- $t_{10} = \dfrac{\sigma_{\delta_1\delta_0}}{t_{00}} = \dfrac{\sigma_{\delta_1\delta_0}}{\sigma_{\delta_0}}$

  $\vdots$

  $t_{R0} = \dfrac{\sigma_{\delta_R\delta_0}}{t_{00}} = \dfrac{\sigma_{\delta_R\delta_0}}{\sigma_{\delta_0}}$
- $t_{11} = \sqrt{\sigma^2_{\delta_1} - \left(t^2_{10}\right)} = \sqrt{\sigma^2_{\delta_1} - \dfrac{\sigma^2_{\delta_1\delta_0}}{\sigma^2_{\delta_0}}} = \sqrt{\dfrac{\sigma^2_{\delta_1}\sigma^2_{\delta_0} - \sigma^2_{\delta_1\delta_0}}{\sigma^2_{\delta_0}}}$

  $\vdots$

  $t_{RR} = \sqrt{\sigma^2_{\delta_R} - \left(t^2_{R0} + t^2_{R1} + \cdots + t^2_{R,R-1}\right)} = \sqrt{\dfrac{\left(\sigma^2_{\delta_1}\right)\left(\sigma^2_{\delta_0}\right) - \left(\sigma^2_{\delta_R\delta_0} + \sigma^2_{\delta_R\delta_1} + \cdots + \sigma^2_{\delta_R\delta_{R-1}}\right)}{\sigma^2_{\delta_0}}}$

  $\vdots$
- $t_{R1} = \dfrac{\sigma_{\delta_0\delta_1} - \left(t_{R0}t_{10}\right)}{t_{11}} = \dfrac{\sigma_{\delta_R\delta_1} - \left(\dfrac{\sigma_{\delta_R\delta_1}\sigma_{\delta_2\delta_0}}{\sigma^2_{\delta_0}}\right)}{\sqrt{\dfrac{\left(\sigma^2_{\delta_1}\right)\left(\sigma^2_{\delta_0}\right) - \sigma^2_{\delta_1\delta_0}}{\sigma^2_{\delta_0}}}}$
- $t_{12} = 0,\ \cdots,\ t_{1r} = 0$

$$t_{12} = 0, \ \cdots, \ t_{1R} = 0$$

$$\vdots$$

$$t_{R-1,R} = 0$$

Sehingga diperoleh matriks T sebagai berikut:

$$\mathbf{T} = \begin{pmatrix} \sigma_{\delta_0} & 0 & \cdots & 0 \\ \dfrac{\sigma_{\delta_1\delta_0}}{\sigma_{\delta_0}} & \sqrt{\dfrac{\sigma_{\delta_1}^2\sigma_{\delta_0}^2 - \sigma_{\delta_1\delta_0}^2}{\sigma_{\delta_0}^2}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \dfrac{\sigma_{\delta_R\delta_0}}{\sigma_{\delta_0}} & \dfrac{\sigma_{\delta_R\delta_1} - \left(\dfrac{\sigma_{\delta_R\delta_0}\sigma_{\delta_1\delta_0}}{\sigma_{\delta_0}^2}\right)}{\sqrt{\dfrac{\left(\sigma_{\delta_1}^2\right)\left(\sigma_{\delta_0}^2\right) - \sigma_{\delta_1\delta_0}^2}{\sigma_{\delta_0}^2}}} & \cdots & \sqrt{\dfrac{\left(\sigma_{\delta_1}^2\right)\left(\sigma_{\delta_0}^2\right) - \left(\sigma_{\delta_R\delta_0}^2 + \sigma_{\delta_R\delta_1}^2 + \cdots + \sigma_{\delta_R\delta_{R-1}}^2\right)}{\sigma_{\delta_0}^2}} \end{pmatrix}$$

dan

$$\mathbf{TT}' = \begin{pmatrix} \sigma_{\delta_0} & 0 & \cdots & 0 \\ \dfrac{\sigma_{\delta_1\delta_0}}{\sigma_{\delta_0}} & \sqrt{\dfrac{\sigma_{\delta_1}^2\sigma_{\delta_0}^2 - \sigma_{\delta_1\delta_0}^2}{\sigma_{\delta_0}^2}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \dfrac{\sigma_{\delta_R\delta_0}}{\sigma_{\delta_0}} & \dfrac{\sigma_{\delta_R\delta_1} - \left(\dfrac{\sigma_{\delta_R\delta_0}\sigma_{\delta_1\delta_0}}{\sigma_{\delta_0}^2}\right)}{\sqrt{\dfrac{\left(\sigma_{\delta_1}^2\right)\left(\sigma_{\delta_0}^2\right) - \sigma_{\delta_1\delta_0}^2}{\sigma_{\delta_0}^2}}} & \cdots & \sqrt{\dfrac{\left(\sigma_{\delta_1}^2\right)\left(\sigma_{\delta_0}^2\right) - \left(\sigma_{\delta_R\delta_0}^2 + \sigma_{\delta_R\delta_1}^2 + \cdots + \sigma_{\delta_R\delta_{R-1}}^2\right)}{\sigma_{\delta_0}^2}} \end{pmatrix} \begin{pmatrix} \sigma_{\delta_0} & \dfrac{\sigma_{\delta_1\delta_0}}{\sigma_{\delta_0}} & \cdots & \dfrac{\sigma_{\delta_R\delta_0}}{\sigma_{\delta_0}} \\ 0 & \sqrt{\dfrac{\sigma_{\delta_1}^2\sigma_{\delta_0}^2 - \sigma_{\delta_1\delta_0}^2}{\sigma_{\delta_0}^2}} & \cdots & \dfrac{\sigma_{\delta_R\delta_1} - \left(\dfrac{\sigma_{\delta_R\delta_0}\sigma_{\delta_1\delta_0}}{\sigma_{\delta_0}^2}\right)}{\sqrt{\dfrac{\left(\sigma_{\delta_1}^2\right)\left(\sigma_{\delta_0}^2\right) - \sigma_{\delta_1\delta_0}^2}{\sigma_{\delta_0}^2}}} \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & \sqrt{\dfrac{\left(\sigma_{\delta_1}^2\right)\left(\sigma_{\delta_0}^2\right) - \left(\sigma_{\delta_R\delta_0}^2 + \sigma_{\delta_R\delta_1}^2 + \cdots + \sigma_{\delta_R\delta_{R-1}}^2\right)}{\sigma_{\delta_0}^2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{\delta_0}^2 & \sigma_{\delta_1\delta_0} & \cdots & \sigma_{\delta_R\delta_0} \\ \sigma_{\delta_1\delta_0} & \sigma_{\delta_1}^2 & \cdots & \sigma_{\delta_R\delta_1} \\ \vdots & \vdots & & \vdots \\ \sigma_{\delta_R\delta_0} & \sigma_{\delta_R\delta_1} & \cdots & \sigma_{\delta_R}^2 \end{pmatrix}$$

Karena $\sigma_{\delta_i\delta_j} = \sigma_{\delta_j\delta_i}$, $i; j = 0,1,...,R$, maka matriks di atas dapat juga ditulis sebagai:

$$\mathbf{TT'} = \begin{pmatrix} \sigma^2_{\delta_0} & \sigma_{\delta_1\delta_0} & \cdots & \sigma_{\delta_R\delta_0} \\ \sigma_{\delta_1\delta_0} & \sigma^2_{\delta_1} & \cdots & \sigma_{\delta_R\delta_1} \\ \vdots & \vdots & & \vdots \\ \sigma_{\delta_R\delta_0} & \sigma_{\delta_R\delta_1} & \cdots & \sigma^2_{\delta_R} \end{pmatrix} = \begin{pmatrix} \sigma^2_{\delta_0} & \sigma_{\delta_0\delta_1} & \cdots & \sigma_{\delta_0\delta_R} \\ \sigma_{\delta_1\delta_0} & \sigma^2_{\delta_1} & \cdots & \sigma_{\delta_1\delta_R} \\ \vdots & \vdots & & \vdots \\ \sigma_{\delta_R\delta_0} & \sigma_{\delta_R\delta_1} & \cdots & \sigma^2_{\delta_R} \end{pmatrix} = \mathbf{\Sigma_\delta}$$

LAMPIRAN 3

Membuktikan variabel $\boldsymbol{\theta}$, $\boldsymbol{\theta} = \mathbf{T}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu})$ berdistribusi $N_{R+1}(\mathbf{0,I})$ Jika diketahui $\boldsymbol{\beta}$ berdistribusi $N_{R+1}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{\boldsymbol{\delta}})$

Misalkan $\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_R \end{pmatrix}$ adalah suatu variabel random dari distribusi *multivariate* normal dengan

Mean: $\boldsymbol{\mu} = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_R \end{pmatrix}$ dan covariansi matrix : $\boldsymbol{\Sigma}_{\boldsymbol{\delta}} = \begin{pmatrix} \sigma^2_{\delta_0} & \sigma_{\delta_0\delta_1} & \cdots & \sigma_{\delta_0\delta_R} \\ \sigma_{\delta_1\delta_0} & \sigma^2_{\delta_1} & \cdots & \sigma_{\delta_1\delta_R} \\ \vdots & \vdots & & \vdots \\ \sigma_{\delta_R\delta_0} & \sigma_{\delta_R\delta_1} & \cdots & \sigma^2_{\delta_R} \end{pmatrix}$

Perhatikan transformasi:

$\boldsymbol{\theta} = \mathbf{T}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu})$, dimana $\mathbf{TT}' = \boldsymbol{\Sigma}_{\boldsymbol{\delta}}$ adalah dekomposisi *Cholesky* dari $\boldsymbol{\Sigma}_{\boldsymbol{\delta}}$.

Akan dibuktikan bahwa variabel $\boldsymbol{\theta}$ berdistribusi *multivariate* normal standar $N_{R+1}(\mathbf{0,I})$.

<u>Bukti:</u>

Diketahui bahwa : $E(\boldsymbol{\beta}) = \boldsymbol{\mu}$, dan

$$\text{cov}(\boldsymbol{\beta}) = E\left[(\boldsymbol{\beta} - \boldsymbol{\mu})(\boldsymbol{\beta} - \boldsymbol{\mu})'\right] = \boldsymbol{\Sigma}_{\boldsymbol{\delta}}$$

Maka,

$$E(\boldsymbol{\theta}) = E\left(\mathbf{T}^{-1}(\boldsymbol{\beta}-\boldsymbol{\mu})\right)$$
$$= \mathbf{T}^{-1}E(\boldsymbol{\beta}-\boldsymbol{\mu})$$
$$= \mathbf{0}; \quad \text{karena } E(\boldsymbol{\beta}) = \boldsymbol{\mu}$$

dan,

$$\text{cov}(\boldsymbol{\theta}) = E(\boldsymbol{\theta}\boldsymbol{\theta}') - \left[E(\boldsymbol{\theta})\right]^2$$
$$= E(\boldsymbol{\theta}\boldsymbol{\theta}'); \qquad \text{karena } E(\boldsymbol{\theta}) = \mathbf{0}$$
$$= E\left(\mathbf{T}^{-1}(\boldsymbol{\beta}-\boldsymbol{\mu})\left(\mathbf{T}^{-1}(\boldsymbol{\beta}-\boldsymbol{\mu})\right)'\right)$$
$$= \mathbf{T}^{-1}E\left((\boldsymbol{\beta}-\boldsymbol{\mu})(\boldsymbol{\beta}-\boldsymbol{\mu})'\left(\mathbf{T}^{-1}\right)'\right)$$
$$= \mathbf{T}^{-1}E\left((\boldsymbol{\beta}-\boldsymbol{\mu})(\boldsymbol{\beta}-\boldsymbol{\mu})'\right)\left(\mathbf{T}^{-1}\right)'$$
$$= \mathbf{T}^{-1}\ \boldsymbol{\Sigma}_{\boldsymbol{\delta}}\ \left(\mathbf{T}^{-1}\right)'$$
$$= \mathbf{T}^{-1}(\mathbf{T}\mathbf{T}')\left(\mathbf{T}^{-1}\right)'$$
$$= \left(\mathbf{T}^{-1}\mathbf{T}\right)\left(\mathbf{T}^{-1}\mathbf{T}\right)'$$
$$= \mathbf{I}$$

Terbukti bahwa variabel $\boldsymbol{\theta}$ berdistribusi *multivariate* normal standar dengan mean **0** dan matriks kovariansi **I** (matriks identitas), dinotasikan $N_{R+1}(\mathbf{0},\mathbf{I})$.

LAMPIRAN 4

Metode *Newton Raphson*

Misalkan $\mathbf{\Theta}$ adalah vektor dari parameter-parameter yang tidak diketahui, yaitu :

$$\mathbf{\Theta} = \left(\mu_0,\ldots,\mu_R \quad \alpha_1,\ldots,\alpha_p \quad \gamma_2,\ldots,\gamma_{C-1} \quad t_{00},\ldots,t_{RR}\right)' \qquad (1)$$

Misalkan pula, $l(\mathbf{\Theta})$ adalah fungsi *marginal log-likelihood* dari vektor $\mathbf{\Theta}$, $U(\mathbf{\Theta})$ adalah vektor turunan parsial pertama dari $l(\mathbf{\Theta})$, yaitu

$$U(\mathbf{\Theta}) = \begin{bmatrix} U_{\mu_0}(\mathbf{\Theta}) \\ \vdots \\ U_{\mu_R}(\mathbf{\Theta}) \\ U_{\alpha_1}(\mathbf{\Theta}) \\ \vdots \\ U_{\alpha_P}(\mathbf{\Theta}) \\ U_{\gamma_2}(\mathbf{\Theta}) \\ \vdots \\ U_{\gamma_{C-1}}(\mathbf{\Theta}) \\ U_{t_{00}}(\mathbf{\Theta}) \\ \vdots \\ U_{t_{RR}}(\mathbf{\Theta}) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial l(\mathbf{\Theta})}{\partial \mu_0} \\ \vdots \\ \dfrac{\partial l(\mathbf{\Theta})}{\partial \mu_R} \\ \dfrac{\partial l(\mathbf{\Theta})}{\partial \alpha_1} \\ \vdots \\ \dfrac{\partial l(\mathbf{\Theta})}{\partial \alpha_P} \\ \dfrac{\partial l(\mathbf{\Theta})}{\partial \gamma_2} \\ \vdots \\ \dfrac{\partial l(\mathbf{\Theta})}{\partial \gamma_{C-1}} \\ \dfrac{\partial l(\mathbf{\Theta})}{\partial t_{00}} \\ \vdots \\ \dfrac{\partial l(\mathbf{\Theta})}{\partial t_{RR}} \end{bmatrix}$$

$H(\mathbf{\Theta})$ adalah matriks turunan parsial kedua dari $l(\mathbf{\Theta})$, yaitu:

$$
H(\mathbf{\Theta}) = \frac{\partial U(\mathbf{\Theta})}{\partial \mathbf{\Theta}'}
$$

$$
= \begin{bmatrix} \frac{\partial U(\mathbf{\Theta})}{\partial \mu_0} & \cdots & \frac{\partial U(\mathbf{\Theta})}{\partial \mu_R} & \frac{\partial U(\mathbf{\Theta})}{\partial \alpha_1} & \cdots & \frac{\partial U(\mathbf{\Theta})}{\partial \alpha_p} & \frac{\partial U(\mathbf{\Theta})}{\partial \gamma_2} & \cdots & \frac{\partial U(\mathbf{\Theta})}{\partial \gamma_{C-1}} & \frac{\partial U(\mathbf{\Theta})}{\partial t_{00}} & \cdots & \frac{\partial U(\mathbf{\Theta})}{\partial t_{RR}} \end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0^2} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_R \partial \mu_0} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_1 \partial \mu_0} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P \partial \mu_0} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2 \partial \mu_0} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1} \partial \mu_0} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{00} \partial \mu_0} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial \mu_0} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0 \partial \mu_R} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_R^2} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_R \partial \mu_R} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P \partial \mu_R} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2 \partial \mu_R} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1} \partial \mu_R} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial \mu_R} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial \mu_R} \\
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0 \partial \alpha_1} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_R \partial \alpha_1} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_1^2} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P \partial \alpha_1} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2 \partial \alpha_1} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1} \partial \alpha_1} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{00} \partial \alpha_1} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial \alpha_1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0 \partial \alpha_P} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_R \partial \alpha_P} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_1 \partial \alpha_P} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P^2} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2 \partial \alpha_P} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1} \partial \alpha_P} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{00} \partial \alpha_P} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial \alpha_P} \\
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0 \partial \gamma_2} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_R \partial \gamma_2} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_1 \partial \gamma_2} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P \partial \gamma_2} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2^2} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1} \partial \gamma_2} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{00} \partial \gamma_2} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial \gamma_2} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0 \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_R \partial \gamma_{C-1}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_1 \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P \partial \gamma_{C-1}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2 \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1}^2} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{00} \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial \gamma_{C-1}} \\
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0 \partial t_{11}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_R \partial t_{00}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_1 \partial t_{00}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P \partial t_{00}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2 \partial t_{00}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1} \partial t_{00}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{00}^2} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR} \partial t_{11}} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_0 \partial t_{RR}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \mu_{RR} \partial t_{RR}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_1 \partial t_{RR}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \alpha_P \partial t_{RR}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_2 \partial t_{RR}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial \gamma_{C-1} \partial t_{RR}} & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{00} \partial t_{RR}} & \cdots & \frac{\partial^2 l(\mathbf{\Theta})}{\partial t_{RR}^2}
\end{bmatrix}
$$

Dengan menggunakan metode *Newton Raphson*, akan dicari taksiran dari $\mathbf{\Theta}$ yang merupakan penyelesaian dari $U(\mathbf{\Theta}) = \mathbf{0}$. Jika diberi taksiran awal dari $\mathbf{\Theta}$, yaitu $\hat{\mathbf{\Theta}}^{(0)}$, maka dapat diperoleh pendekatan deret Taylor orde pertama dari vektor $U(\mathbf{\Theta})$ di sekitar $\mathbf{\Theta} = \hat{\mathbf{\Theta}}^{(0)}$, yaitu:

$$U(\boldsymbol{\Theta}) \approx U\left(\hat{\boldsymbol{\Theta}}^{(0)}\right) + \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial \mu_0}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(\mu_0 - \hat{\mu}_0^{(0)}\right) + \cdots + \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial \mu_R}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(\mu_r - \hat{\mu}_r^{(0)}\right) + \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial \alpha_1}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(\alpha_1 - \hat{\alpha}_1^{(0)}\right) + \cdots + \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial \alpha_P}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(\alpha_P - \hat{\alpha}_P^{(0)}\right)$$

$$+ \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial \gamma_2}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(\gamma_2 - \hat{\gamma}_2^{(0)}\right) + \cdots + \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial \gamma_{C-1}}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(\gamma_{C-1} - \hat{\gamma}_{C-1}^{(0)}\right) + \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial t_{00}}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(t_{00} - \hat{t}_{00}^{(0)}\right) + \cdots + \left.\frac{\partial U(\boldsymbol{\Theta})}{\partial t_{RR}}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(t_{RR} - \hat{t}_{RR}^{(0)}\right)$$

$$\approx U\left(\hat{\boldsymbol{\Theta}}^{(0)}\right) + \left.\begin{bmatrix} \frac{\partial U(\boldsymbol{\Theta})}{\partial \mu_0} & \cdots & \frac{\partial U(\boldsymbol{\Theta})}{\partial \mu_R} & \frac{\partial U(\boldsymbol{\Theta})}{\partial \alpha_1} & \cdots & \frac{\partial U(\boldsymbol{\Theta})}{\partial \alpha_P} & \frac{\partial U(\boldsymbol{\Theta})}{\partial \gamma_2} & \cdots & \frac{\partial U(\boldsymbol{\Theta})}{\partial \gamma_{C-1}} & \frac{\partial U(\boldsymbol{\Theta})}{\partial t_{00}} & \cdots & \frac{\partial U(\boldsymbol{\Theta})}{\partial t_{RR}} \end{bmatrix}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \begin{bmatrix} \left(\mu_0 - \hat{\mu}_0^{(0)}\right) \\ \vdots \\ \left(\mu_r - \hat{\mu}_r^{(0)}\right) \\ \left(\alpha_1 - \hat{\alpha}_1^{(0)}\right) \\ \vdots \\ \left(\alpha_P - \hat{\alpha}_P^{(0)}\right) \\ \left(\gamma_2 - \hat{\gamma}_2^{(0)}\right) \\ \vdots \\ \left(\gamma_{C-1} - \hat{\gamma}_{C-1}^{(0)}\right) \\ \left(t_{00} - \hat{t}_{00}^{(0)}\right) \\ \vdots \\ \left(t_{RR} - \hat{t}_{RR}^{(0)}\right) \end{bmatrix}$$

$$\approx U\left(\hat{\boldsymbol{\Theta}}^{(0)}\right) + \left.\begin{bmatrix}
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_R \partial \mu_0} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_1 \partial \mu_0} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P \partial \mu_0} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2 \partial \mu_0} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1} \partial \mu_0} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00} \partial \mu_0} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial \mu_0} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0 \partial \mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_R^2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_R \partial \mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P \partial \mu_R} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2 \partial \mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1} \partial \mu_R} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial \mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial \mu_R} \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0 \partial \alpha_1} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_R \partial \alpha_1} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_1^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P \partial \alpha_1} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2 \partial \alpha_1} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1} \partial \alpha_1} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00} \partial \alpha_1} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial \alpha_1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0 \partial \alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_R \partial \alpha_P} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_1 \partial \alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P^2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2 \partial \alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1} \partial \alpha_P} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00} \partial \alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial \alpha_P} \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0 \partial \gamma_2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_R \partial \gamma_2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_1 \partial \gamma_2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P \partial \gamma_2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1} \partial \gamma_2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00} \partial \gamma_2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial \gamma_2} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0 \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_R \partial \gamma_{C-1}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_1 \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P \partial \gamma_{C-1}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2 \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1}^2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00} \partial \gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial \gamma_{C-1}} \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0 \partial t_{11}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_R \partial t_{00}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_1 \partial t_{00}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P \partial t_{00}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2 \partial t_{00}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1} \partial t_{00}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR} \partial t_{11}} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_0 \partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \mu_{RR} \partial t_{RR}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_1 \partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha_P \partial t_{RR}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_2 \partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{C-1} \partial t_{RR}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00} \partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}^2}
\end{bmatrix}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}^{(0)}} \left(\boldsymbol{\Theta} - \hat{\boldsymbol{\Theta}}^{(0)}\right)$$

$$\approx U\left(\hat{\boldsymbol{\Theta}}^{(0)}\right) + H\left(\hat{\boldsymbol{\Theta}}^{(0)}\right)\left(\boldsymbol{\Theta} - \hat{\boldsymbol{\Theta}}^{(0)}\right)$$

dengan,

$U\left(\hat{\mathbf{\Theta}}^{(0)}\right)$ adalah vektor turunan parsial pertama dari $l(\mathbf{\Theta})$ dihitung pada $\mathbf{\Theta} = \hat{\mathbf{\Theta}}^{(0)}$.

$H\left(\hat{\mathbf{\Theta}}^{(0)}\right)$ adalah matriks turunan parsial kedua dari $l(\mathbf{\Theta})$ dihitung pada $\mathbf{\Theta} = \hat{\mathbf{\Theta}}^{(0)}$.

Karena ingin dicari penyelesaian dari $U(\mathbf{\Theta}) = \mathbf{0}$, maka dapat ditulis:

$$\mathbf{0} = U(\mathbf{\Theta}) \approx U\left(\hat{\mathbf{\Theta}}^{(0)}\right) + H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}^{(0)}\right)$$

$$H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}^{(0)}\right) = -U\left(\hat{\mathbf{\Theta}}^{(0)}\right)$$

Jika matriks $H\left(\hat{\mathbf{\Theta}}^{(0)}\right)$ nonsingular maka,

$$\left[H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\right]^{-1} H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}^{(0)}\right) = -\left[H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\right]^{-1} U\left(\hat{\mathbf{\Theta}}^{(0)}\right)$$

$$\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}^{(0)}\right) = -\left[H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\right]^{-1} U\left(\hat{\mathbf{\Theta}}^{(0)}\right)$$

$$\mathbf{\Theta} = \hat{\mathbf{\Theta}}^{(0)} - \left[H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\right]^{-1} U\left(\hat{\mathbf{\Theta}}^{(0)}\right)$$

Nilai $\mathbf{\Theta}$ kemudian menjadi taksiran baru yang kemudian dinotasikan dengan $\hat{\mathbf{\Theta}}^{(1)}$, yaitu:

$$\hat{\mathbf{\Theta}}^{(1)} = \hat{\mathbf{\Theta}}^{(0)} - \left[H\left(\hat{\mathbf{\Theta}}^{(0)}\right)\right]^{-1} U\left(\hat{\mathbf{\Theta}}^{(0)}\right)$$

Jika $\hat{\mathbf{\Theta}}^{(1)} \approx \hat{\mathbf{\Theta}}^{(0)}$ (misalkan $\left\|\hat{\mathbf{\Theta}}^{(1)} - \hat{\mathbf{\Theta}}^{(0)}\right\| < 10^{-4}$), maka hentikan proses iterasi dan kemudian ambil $\hat{\mathbf{\Theta}}^{(1)}$ sebagai taksiran dari $\mathbf{\Theta}$. Sebaliknya, lanjutkan iterasi. Jika diberikan taksiran $\hat{\mathbf{\Theta}}^{(1)}$, maka dapat diperoleh pendekatan deret Taylor orde pertama dari vektor $U(\mathbf{\Theta})$ sekitar $\mathbf{\Theta} = \hat{\mathbf{\Theta}}^{(1)}$, yaitu:

$$U(\mathbf{\Theta}) \approx U\left(\hat{\mathbf{\Theta}}^{(1)}\right) + H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}^{(1)}\right)$$

dengan:

$U\left(\hat{\mathbf{\Theta}}^{(1)}\right)$ adalah vektor turunan parsial pertama dari $l\left(\mathbf{\Theta}\right)$ dihitung pada $\mathbf{\Theta}=\hat{\mathbf{\Theta}}^{(1)}$.

$H\left(\hat{\mathbf{\Theta}}^{(1)}\right)$ adalah matriks turunan parsial kedua dari $l\left(\mathbf{\Theta}\right)$ dihitung pada $\mathbf{\Theta}=\hat{\mathbf{\Theta}}^{(1)}$.

Dengan menyelesaikan ,

$$\mathbf{0}=U\left(\mathbf{\Theta}\right)\approx U\left(\hat{\mathbf{\Theta}}^{(1)}\right)+H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}^{(1)}\right)$$

diperoleh
$$H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}^{(1)}\right)=-U\left(\hat{\mathbf{\Theta}}^{(1)}\right)$$

Jika matriks $H\left(\hat{\mathbf{\Theta}}^{(1)}\right)$ nonsingular, maka:

$$\left[H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\right]^{-1}H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}^{(1)}\right)=-\left[H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\right]^{-1}U\left(\hat{\mathbf{\Theta}}^{(1)}\right)$$

$$\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}^{(1)}\right)=-\left[H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\right]^{-1}U\left(\hat{\mathbf{\Theta}}^{(1)}\right)$$

$$\mathbf{\Theta}=\hat{\mathbf{\Theta}}^{(1)}-\left[H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\right]^{-1}U\left(\hat{\mathbf{\Theta}}^{(1)}\right)$$

Nilai $\mathbf{\Theta}$ kemudian menjadi taksiran baru yang kemudian dinotasikan dengan $\hat{\mathbf{\Theta}}^{(2)}$, yaitu:

$$\hat{\mathbf{\Theta}}^{(2)}=\hat{\mathbf{\Theta}}^{(1)}-\left[H\left(\hat{\mathbf{\Theta}}^{(1)}\right)\right]^{-1}U\left(\hat{\mathbf{\Theta}}^{(1)}\right)$$

Dengan mengulang langkah yang sama dapat diperoleh taksiran $\hat{\mathbf{\Theta}}^{(3)}$, $\hat{\mathbf{\Theta}}^{(4)}$, dan seterusnya.

Secara umum, dapat ditentukan taksiran dari $\mathbf{\Theta}$ pada iterasi ke m+1, yaitu $\hat{\mathbf{\Theta}}^{(m+1)}$, secara iteratif menggunakan algoritma:

$$\hat{\mathbf{\Theta}}^{(m+1)}=\hat{\mathbf{\Theta}}^{(m)}-\left[H\left(\hat{\mathbf{\Theta}}^{(m)}\right)\right]^{-1}U\left(\hat{\mathbf{\Theta}}^{(m)}\right),\quad m = 0, 1, ...$$

Dengan:

$U\left(\hat{\mathbf{\Theta}}^{(m)}\right)$ adalah vektor turunan parsial pertama dari $l(\mathbf{\Theta})$ dihitung pada $\mathbf{\Theta} = \hat{\mathbf{\Theta}}^{(m)}$.

$H\left(\hat{\mathbf{\Theta}}^{(m)}\right)$ adalah matriks turunan parsial kedua dari $l(\mathbf{\Theta})$ dihitung pada $\mathbf{\Theta} = \hat{\mathbf{\Theta}}^{(m)}$.

Jika $\hat{\mathbf{\Theta}}^{m+1} \approx \hat{\mathbf{\Theta}}^{(m)}$ (misalkan $\left\|\hat{\mathbf{\Theta}}^{(m+1)} - \hat{\mathbf{\Theta}}^{(m)}\right\| < 10^{-4}$), maka hentikan proses iterasi dan kemudian ambil $\hat{\mathbf{\Theta}}^{(m+1)}$ sebagai taksiran dari $\mathbf{\Theta}$.

LAMPIRAN 5

Menunjukkan $E\left[H\left(\hat{\mathbf{\Theta}}^{(m)}\right)\right] = -\sum_{i=1}^{N} \frac{1}{\mathrm{h}\left(\mathbf{Y}_i\right)^2} \cdot \left(\frac{\partial \mathrm{h}\left(\mathbf{Y}_i\right)}{\partial \hat{\mathbf{\Theta}}^{(m)}}\right) \cdot \left(\frac{\partial \mathrm{h}\left(\mathbf{Y}_i\right)}{\partial \hat{\mathbf{\Theta}}^{(m)}}\right)'$

Untuk tujuan tersebut mula-mula dimisalkan terdapat N observasi yang saling bebas, dan $\mathbf{Y}_i$ merupakan vektor dari variabel respon ordinal untuk sembarang observasi ke-*i*, *i* = 1,2, ..., N.

$h\left(\mathbf{Y}_i\right)$ merupakan pdf. marginal dari $\mathbf{Y}_i$ dalam populasi, dari persamaan (3.2.1.40) pada bab 3, diketahui:

$$\mathrm{h}(\mathbf{Y}_i) = \int_{\boldsymbol{\theta}} \ell\left(\mathbf{Y}_i \,|\, \boldsymbol{\theta}, \boldsymbol{\alpha}\right) g(\boldsymbol{\theta})\, d\boldsymbol{\theta}$$

Fungsi marginal likelihood dari N unit level-2 adalah:

$$\mathrm{L}\left(\mathbf{\Theta}\right) = \prod_{i=1}^{N} h(\mathbf{Y}_i)$$

Misalkan $l\left(\mathbf{\Theta}\right)$ adalah fungsi *log likelihood* dari vektor $\mathbf{\Theta}$,

$$l\left(\mathbf{\Theta}\right) = \log\ \mathrm{L} = \sum_{i=1}^{N} \log h(\mathbf{Y}_i)$$

$U\left(\mathbf{\Theta}\right)$ adalah vektor turunan parsial pertama dari $l\left(\mathbf{\Theta}\right)$ seperti yang diberikan pada lampiran 4.

$H\left(\mathbf{\Theta}\right)$ adalah matriks turunan parsial kedua dari $l\left(\mathbf{\Theta}\right)$ seperti yang diberikan pada lampiran 4.

Ekspektasi dari matriks $H\left(\mathbf{\Theta}\right)$ dinyatakan dengan,

$$E\left[H(\boldsymbol{\Theta})\right]=E\begin{bmatrix}
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\mu_0} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\mu_0} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\mu_0} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\mu_0} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\mu_0} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\mu_0} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\mu_0} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R^2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_R\partial\mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\mu_R} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\mu_R} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\mu_R} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\mu_R} \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\alpha_1} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\alpha_1} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\alpha_1} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\alpha_1} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\alpha_1} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\alpha_1} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\alpha_1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\alpha_P} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P^2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\alpha_P} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\alpha_P} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\alpha_P} \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\gamma_2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\gamma_2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\gamma_2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\gamma_2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\gamma_2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\gamma_2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\gamma_2} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\gamma_{C-1}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\gamma_{C-1}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}^2} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\gamma_{C-1}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\gamma_{C-1}} \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial t_{11}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial t_{00}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial t_{00}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial t_{00}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial t_{00}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial t_{00}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}^2} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial t_{11}} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_{RR}\partial t_{RR}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial t_{RR}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial t_{RR}} & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial t_{RR}} & \cdots & \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}^2}
\end{bmatrix}$$

$$E[H(\boldsymbol{\Theta})]=\begin{bmatrix}
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0^2}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\mu_0}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\mu_0}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\mu_0}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\mu_0}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\mu_0}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\mu_0}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\mu_0}\right) \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\mu_R}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R^2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\mu_R}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\mu_R}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\mu_R}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\mu_R}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\mu_R}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\mu_R}\right) \\
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\alpha_1}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\alpha_1}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1^2}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\alpha_1}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\alpha_1}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\alpha_1}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\alpha_1}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\alpha_1}\right) \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\alpha_P}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\alpha_P}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\alpha_P}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P^2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\alpha_P}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\alpha_P}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\alpha_P}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\alpha_P}\right) \\
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\gamma_2}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\gamma_2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\gamma_2}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\gamma_2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2^2}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial\gamma_2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\gamma_2}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\gamma_2}\right) \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial\gamma_{C-1}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial\gamma_{C-1}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial\gamma_{C-1}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial\gamma_{C-1}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial\gamma_{C-1}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}^2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial\gamma_{C-1}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial\gamma_{C-1}}\right) \\
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial t_{11}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial t_{00}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial t_{00}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial t_{00}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial t_{00}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial t_{00}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}^2}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}\partial t_{00}}\right) \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_0\partial t_{RR}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_R\partial t_{RR}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_1\partial t_{RR}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_P\partial t_{RR}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_2\partial t_{RR}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{C-1}\partial t_{RR}}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{00}\partial t_{RR}}\right) & \cdots & E\left(\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{RR}^2}\right)
\end{bmatrix}$$

- Ekspektasi turunan parsial kedua dari $l(\boldsymbol{\Theta})$ terhadap nilai tertentu

  $\mu_{r'}$ dan $\mu_{r''}$ $r', r'' = 0,1,...,R$ adalah

$$E\left[\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_r \partial\mu_{r'}}\right] = E\left[\frac{\partial^2\left(\sum_{i=1}^{N}\log h(\mathbf{Y}_i)\right)}{\partial\mu_r\partial\mu_{r'}}\right] = E\left[\sum_{i=1}^{N}\left(\frac{\partial^2 \log h(\mathbf{Y}_i)}{\partial\mu_r\partial\mu_{r'}}\right)\right]$$

$$= \sum_{i=1}^{N} E\left[\frac{\partial^2 \log h(\mathbf{Y}_i)}{\partial\mu_r\partial\mu_{r''}}\right] = \sum_{i=1}^{N} E\left[\frac{\partial}{\partial\mu_r}\left(\frac{\partial \log h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right]$$

$$= \sum_{i=1}^{N} E\left[\frac{\partial}{\partial\mu_r}\left(\frac{1}{h(\mathbf{Y}_i)}\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right]$$

$$= \sum_{i=1}^{N} E\left[\left(\frac{\partial}{\partial\mu_r}\cdot\frac{1}{h(\mathbf{Y}_i)}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right]$$

$$= \sum_{i=1}^{N} E\left[-\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right) + \frac{1}{h(\mathbf{Y}_i)}\frac{\partial^2 h(\mathbf{Y}_i)}{\partial\mu_r\partial\mu_{r'}}\right]$$

$$= \sum_{i=1}^{N} \left[E\left(-\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right) + E\left(\frac{1}{h(\mathbf{Y}_i)}\frac{\partial^2 h(\mathbf{Y}_i)}{\partial\mu_r\partial\mu_{r'}}\right)\right]$$

$$= \sum_{i=1}^{N} \left\{-E\left(\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right) + \left[\sum_{\mathbf{Y}_i}\left(\frac{1}{h(\mathbf{Y}_i)}\frac{\partial^2 h(\mathbf{Y}_i)}{\partial\mu_r\partial\mu_{r'}}\right)\cdot h(\mathbf{Y}_i)\right]\right\}$$

$$= \sum_{i=1}^{N} \left\{-E\left(\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right) + \left[\sum_{\mathbf{Y}_i}\left(\frac{\partial^2 h(\mathbf{Y}_i)}{\partial\mu_r\partial\mu_{r'}}\right)\right]\right\}$$

$$= \sum_{i=1}^{N} \left\{-E\left(\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right) + \left[\frac{\partial^2}{\partial\mu_r\partial\mu_{r'}}\cdot\sum_{\mathbf{Y}_i} h(\mathbf{Y}_i)\right]\right\}$$

$$= \sum_{i=1}^{N} \left\{-E\left(\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right) + \underbrace{\left[\frac{\partial^2}{\partial\mu_r\partial\mu_{r'}}\cdot 1\right]}_{0}\right\}$$

$$= \sum_{i=1}^{N} \left[-E\left(\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right)\right]$$

$$= E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\right]$$

Dengan mengikuti prosedur yang sama seperti pada persamaan (1), maka diperoleh:

- Ekspektasi turunan parsial kedua dari $l(\boldsymbol{\Theta})$ terhadap nilai tertentu α_p dan $\alpha_{p'}$ $p, p' = 1,...,P$ adalah

$$E\left[\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha_p \partial\alpha_{p'}}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\alpha_p}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\alpha_{p'}}\right)\right] \quad (2)$$

- Ekspektasi turunan parsial kedua dari $l(\boldsymbol{\Theta})$ terhadap nilai tertentu $\gamma_{c'}$ dan $\gamma_{c''}$ $c', c'' = 2,...,C-2$ adalah

$$E\left[\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\gamma_{c'}\partial\gamma_{c''}}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\gamma_{c'}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\gamma_{c''}}\right)\right] \quad (3)$$

- Ekspektasi turunan parsial kedua dari $l(\boldsymbol{\Theta})$ terhadap nilai tertentu t_{rh} dan $\gamma_{r'h'}$, $0 \le r \le h \le R$; $0 \le r' \le h' \le R$ adalah

$$E\left[\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial t_{rh}\partial t_{r'h'}}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{rh}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{r'h'}}\right)\right] \quad (4)$$

- Ekspektasi turunan parsial kedua dari $l(\boldsymbol{\Theta})$ terhadap nilai tertentu μ_r dan α_p, $r = 0,1,...,R$; $\text{p} = 1,...,P$ adalah

$$E\left[\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\mu_r\partial\alpha_p}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\alpha_p}\right)\right] \quad (5)$$

(1)

- Ekspektasi turunan parsial kedua dari $l(\mathbf{\Theta})$ terhadap nilai tertentu μ_r dan $\gamma_{c'}$, $r = 0,1,...,R;\ c' = 2,...,C-1$ adalah

$$E\left[\frac{\partial^2 l(\mathbf{\Theta})}{\partial\mu_r\partial\gamma_{c'}}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_r}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\gamma_{c'}}\right)\right] \quad (6)$$

- Ekspektasi turunan parsial kedua dari $l(\mathbf{\Theta})$ terhadap nilai tertentu $\mu_{r'}$ dan t_{rh}, $r' = 0,1,...,R;\ 0 \le r \le h \le R$ adalah

$$E\left[\frac{\partial^2 l(\mathbf{\Theta})}{\partial\mu_{r'}\partial t_{rh}}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\mu_{r'}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{rh}}\right)\right] \quad (7)$$

- Ekspektasi turunan parsial kedua dari $l(\mathbf{\Theta})$ terhadap nilai tertentu α_p dan $\gamma_{c'}$, $p = 1,...,P;\ c' = 2,...,C-1$ adalah

$$E\left[\frac{\partial^2 l(\mathbf{\Theta})}{\partial\alpha_p\partial\gamma_{c'}}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\alpha_p}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\gamma_{c'}}\right)\right] \quad (8)$$

- Ekspektasi turunan parsial kedua dari $l(\mathbf{\Theta})$ terhadap nilai tertentu α_p dan t_{rh}, $p = 1,...,P;\ 0 \le r \le h \le R$ adalah

$$E\left[\frac{\partial^2 l(\mathbf{\Theta})}{\partial\alpha_p\partial t_{rh}}\right] = E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\alpha_p}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{rh}}\right)\right] \quad (9)$$

- Ekspektasi turunan parsial kedua dari $l(\boldsymbol{\Theta})$ terhadap nilai tertentu

  $\gamma_{c'}$ dan t_{rh} , $c' = 2,...,C-1;\ 0 \le r \le h \le R$ adalah

$$E\left[\frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \gamma_{c'} \partial t_{rh}}\right] = E\left[-\sum_{i=1}^{N} \frac{1}{h(\mathbf{Y}_i)^2} \cdot \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{c'}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{rh}}\right)\right] \quad (10)$$

dengan,

- $$\frac{\partial(\mathrm{h}(\mathbf{Y}_i))}{\partial \gamma_{c'}} = \int_{\boldsymbol{\theta}} \left(\sum_{k=1}^{n_i} \sum_{c=1}^{C} d_{ikc} \left(\frac{\phi(\gamma_c - z_{ik})\delta_{cc'} - \phi(\gamma_{c-1} - z_{ik})\delta_{c-1,c'}}{\Phi(\gamma_c - z_{ik}) - \Phi(\gamma_{c-1} - z_{ik})} \right)\right) \ell(\mathbf{Y}_i \mid \boldsymbol{\theta},\boldsymbol{\alpha})\, g(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

  , c’ = 2, ..., C-1

- $$\frac{\partial(\mathrm{h}(\mathbf{Y}_i))}{\partial \mu_{r'}} = \int_{\boldsymbol{\theta}} \left(\sum_{k=1}^{n_i} \sum_{c=1}^{C} d_{ikc} \left(\frac{\phi(\gamma_{c-1} - z_{ik}) - \phi(\gamma_c - z_{ik})}{\Phi(\gamma_c - z_{ik}) - \Phi(\gamma_{c-1} - z_{ik})} \right) x_{r'ik} \right) \ell(\mathbf{Y}_i \mid \boldsymbol{\theta},\ \boldsymbol{\alpha}) g(\boldsymbol{\theta})\ d\boldsymbol{\theta}$$

  , r’ = 0,1, ..., R

- $$\frac{\partial(\mathrm{h}(\mathbf{Y}_i))}{\partial \alpha_{p'}} = \int_{\boldsymbol{\theta}} \left(\sum_{k=1}^{n_i} \sum_{c=1}^{C} d_{ikc} \left(\frac{\phi(\gamma_{c-1} - z_{ik}) - \phi(\gamma_c - z_{ik})}{\Phi(\gamma_c - z_{ik}) - \Phi(\gamma_{c-1} - z_{ik})} \right) w_{p'ik} \right) \ell(\mathbf{Y}_i \mid \boldsymbol{\theta},\ \boldsymbol{\alpha}) g(\boldsymbol{\theta})\ d\boldsymbol{\theta}$$

  , p’ = 1, ..., P

- $$\frac{\partial(\mathrm{h}(\mathbf{Y}_i))}{\partial t_{rh}} = \int_{\boldsymbol{\theta}} \left(\sum_{k=1}^{n_i} \sum_{c=1}^{C} d_{ikc} \left(\frac{\phi(\gamma_{c-1} - z_{ik}) - \phi(\gamma_c - z_{ik})}{\Phi(\gamma_c - z_{ik}) - \Phi(\gamma_{c-1} - z_{ik})} \right) \theta_h x_{rik} \right) \ell(\mathbf{Y}_i \mid \boldsymbol{\theta},\ \boldsymbol{\alpha}) g(\boldsymbol{\theta})\ d\boldsymbol{\theta}$$

  $0 \le r \le h \le R$

Sehingga, dari persamaan (1), (2), ..., (10) di atas diperoleh:

$$
E[H(\boldsymbol{\Theta})]=E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}
\begin{pmatrix}
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right) \\
\vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right) \\
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right) \\
\vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right) \\
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right) \\
\vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right) \\
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right) \\
\vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\
\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right) & \left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\cdots\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)\left(\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}\right)
\end{pmatrix}\right]
$$

$$
E[H(\boldsymbol{\Theta})]=E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}
\begin{pmatrix}
\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0} \\
\\
\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R} \\
\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1} \\
\vdots \\
\frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P} \\
\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2} \\
\vdots \\
\frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}} \\
\frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}} \\
\vdots \\
\frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}
\end{pmatrix}
\cdot
\begin{pmatrix}
\frac{\partial h(\mathbf{Y}_i)}{\partial \mu_0} & \cdots & \frac{\partial h(\mathbf{Y}_i)}{\partial \mu_R} & \frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_1} & \cdots & \frac{\partial h(\mathbf{Y}_i)}{\partial \alpha_P} & \frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_2} & \cdots & \frac{\partial h(\mathbf{Y}_i)}{\partial \gamma_{C-1}} & \frac{\partial h(\mathbf{Y}_i)}{\partial t_{00}} & \cdots & \frac{\partial h(\mathbf{Y}_i)}{\partial t_{RR}}
\end{pmatrix}
\right]
$$

$$
=E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \boldsymbol{\Theta}}\right)\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial \boldsymbol{\Theta}}\right)'\right]
$$

Jadi, ekspektasi dari taksiran matriks turunan parsial kedua dari $l(\mathbf{\Theta})$ pada iterasi ke-m, m = 0,1,2,... yaitu:

$$E\left[H\left(\hat{\mathbf{\Theta}}^{(m)}\right)\right]=E\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\hat{\mathbf{\Theta}}^{(m)}}\right)\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\hat{\mathbf{\Theta}}^{(m)}}\right)'\right] \quad (11)$$

Karena h($\mathbf{Y}_i$) merupakan fungsi dari

$\mathbf{\Theta}=\left[\mu_0,\ldots,\mu_R \quad \alpha_1,\ldots,\alpha_P \quad \gamma_2,\ldots,\gamma_{C-1} \quad t_{00},\ldots,t_{RR}\right]'$ yang nilainya diketahui, maka

$\left[-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\hat{\mathbf{\Theta}}^{(m)}}\right)\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\hat{\mathbf{\Theta}}^{(m)}}\right)'\right]$ merupakan suatu konstanta, sehingga

persamaan (11) dapat ditulis:

$$E\left[H\left(\hat{\mathbf{\Theta}}^{(m)}\right)\right]=-\sum_{i=1}^{N}\frac{1}{h(\mathbf{Y}_i)^2}\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\hat{\mathbf{\Theta}}^{(m)}}\right)\cdot\left(\frac{\partial h(\mathbf{Y}_i)}{\partial\hat{\mathbf{\Theta}}^{(m)}}\right)'$$

$\therefore$ Terbukti

LAMPIRAN 6

Data untuk Contoh 4.1

No	Class ID	Post-THKS	Pre-THKS	CC	TV	CC*TV
1	403101	3	2	1	0	0
2	403101	4	4	1	0	0
3	403101	3	4	1	0	0
4	403101	4	3	1	0	0
5	403101	4	3	1	0	0
6	403101	3	4	1	0	0
7	403101	2	2	1	0	0
8	403101	4	4	1	0	0
9	403101	5	5	1	0	0
10	403101	4	3	1	0	0
11	403101	3	3	1	0	0
12	403101	4	3	1	0	0
13	403101	3	1	1	0	0
14	403101	4	2	1	0	0
15	403101	2	2	1	0	0
16	403101	4	1	1	0	0
17	403101	4	4	1	0	0
18	403101	3	3	1	0	0
19	403101	3	0	1	0	0
20	403101	4	3	1	0	0
21	403102	2	0	1	0	0
22	403102	5	1	1	0	0
23	403102	3	5	1	0	0
24	404101	3	1	1	1	1
25	404101	4	2	1	1	1
26	404101	2	4	1	1	1
27	404101	3	3	1	1	1
28	404101	2	1	1	1	1
29	404101	1	1	1	1	1
30	404101	3	2	1	1	1
31	404101	5	0	1	1	1
32	404101	2	1	1	1	1
33	404101	3	2	1	1	1
34	404101	4	2	1	1	1
35	404102	3	1	1	1	1
36	404102	1	1	1	1	1
37	404102	0	0	1	1	1

38	404102	4	4	1	1	1
39	404102	5	1	1	1	1
40	404102	3	2	1	1	1
41	404102	3	1	1	1	1
42	404102	4	2	1	1	1
43	404102	2	2	1	1	1
44	404103	3	2	1	1	1
45	404103	2	1	1	1	1
46	404103	2	2	1	1	1
47	404103	3	1	1	1	1
48	404103	3	1	1	1	1
49	193101	2	1	0	0	0
50	193101	2	3	0	0	0
51	193101	3	0	0	0	0
52	193101	2	3	0	0	0
53	193101	1	1	0	0	0
54	193101	2	2	0	0	0
55	193101	4	3	0	0	0
56	193101	2	3	0	0	0
57	193101	3	3	0	0	0
58	193101	3	1	0	0	0
59	193101	1	5	0	0	0
60	193101	2	1	0	0	0
61	193101	1	3	0	0	0
62	193101	2	2	0	0	0
63	193101	1	0	0	0	0
64	193101	4	3	0	0	0
65	193101	3	4	0	0	0
66	193101	5	2	0	0	0
67	193101	4	2	0	0	0
68	193101	1	3	0	0	0
69	193101	3	2	0	0	0
70	193101	3	3	0	0	0
71	193101	2	1	0	0	0
72	193101	4	2	0	0	0
73	193101	3	0	0	0	0
74	193101	2	3	0	0	0
75	194101	1	3	0	0	0
76	194101	2	1	0	0	0
77	194101	3	2	0	0	0
78	194101	3	2	0	0	0
79	194101	1	0	0	0	0
80	194101	3	0	0	0	0
81	194101	2	4	0	0	0
82	194101	2	2	0	0	0
83	194101	3	1	0	0	0

LAMPIRAN 7

Output MIXOR 2.0 *for Windows* untuk Contoh 4.1

```
MIXOR - The program for mixed-effects ordinal regression analysis
          (version 2)

 TVSFP study - Post-Test Tobacco & Health Knowledge ORDINAL
 Students nested within CLASSROOMS - 1 random effect

 Response function: normal
 Random-effects distribution: normal

 Covariate(s) and random-effect(s) mean subtracted from thresholds
 ==> positive coefficient = positive association between regressor
     and ordinal outcome

 Numbers of observations
 -----------------------

 Level 1 observations =   1600
 Level 2 observations =    135

 The number of level 1 observations per level 2 unit are:

  20   3  11   9   5  26  11  10  15  12  12  10  21  10  17  19   2   4  21
  16  15  13   2  14  13   1  12  18  21  17  16  15  16  21  21  27  17   3
   2  15   7  24  22  15  19   7  12   8   6  11   7   7   8   3   5   8   3
   8   9   8   2  11   9  21  13  12  12  14   9   6  11  10  12  11   6   6
  14  10  14   2   3   2   4   3   6  10  14  11   6  22   4   7  22  18  23
  19  14   5  14  28  15  15  11  12  11  11  15  17  24  20  15   6   8  14
   5  11   9  17  14  11  17  15   6   7  14  10  14  18   4   9   7  12  15
  11  10

 Descriptive statistics for all variables
 ----------------------------------------
 Variable         Minimum          Maximum             Mean      Stand. Dev.

 THKScore         1.00000          4.00000          2.58688          1.11612
```

Intrcpt	1.00000	1.00000	1.00000	0.00000
PreTHKS	0.00000	6.00000	2.06938	1.26018
CC	0.00000	1.00000	0.47688	0.49962
TV	0.00000	1.00000	0.49938	0.50016
CC*TV	0.00000	1.00000	0.23938	0.42684

```
Categories of the response variable THKScore
--------------------------------------------

Category       Frequency       Proportion
    1.00          355.00          0.22188
    2.00          398.00          0.24875
    3.00          400.00          0.25000
    4.00          447.00          0.27938


Starting values
---------------
mean         1.206
covariates  -0.212 -0.451 -0.140  0.201
var. terms   0.316
thresholds   0.711  1.376

==> The number of level 2 observations with non-varying responses
    =      6 (   4.44 percent )


---------------------------------------------------------
* Final Results - Maximum Marginal Likelihood Estimates *
---------------------------------------------------------
Total Iterations  =  15
Quad Pts per Dim  =  10
Log Likelihood    =   -2117.717
Deviance (-2logL) =    4235.433
Ridge             =       0.000
```

Variable	Estimate	Stand. Error	Z	p-value
Intrcpt	0.06220	0.09132	0.68114	0.49578 (2)
PreTHKS	0.24247	0.02351	10.31253	0.00000 (2)
CC	0.50945	0.11225	4.53860	0.00001 (2)
TV	0.12363	0.10032	1.23232	0.21783 (2)
CC*TV	-0.19378	0.15013	-1.29074	0.19679 (2)

```
Random effect variance term (standard deviation)
Intrcpt          0.26163          0.04504           5.80822     0.00000  (1)
Thresholds (for identification: threshold 1 = 0)
     2           0.76063          0.03584          21.22279     0.00000  (1)
     3           1.48749          0.04403          33.78686     0.00000  (1)

note: (1) = 1-tailed p-value
      (2) = 2-tailed p-value

Calculation of the intracluster correlation
-------------------------------------------
residual variance = 1 (assumed)
cluster  variance = (0.262 * 0.262) =  0.068

intracluster correlation =  0.068 / ( 0.068 + 1.000) = 0.064
```

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Data untuk Contoh 4.2

No.	*Patient ID*	*Ordinal IMPS79 Score*	*Drug*	*Time*		*Drug*Time*
				Week	*Sqrt Week*	
1	1103	4	1	0	0	0
2	1103	2	1	1	1	1
3	1103	-9	1	2	1,4142	1,4142
4	1103	2	1	3	1,7321	1,7321
5	1103	-9	1	4	2	2
6	1103	-9	1	5	2,2361	2,2361
7	1103	2	1	6	2,4495	2,4495
8	1104	4	1	0	0	0
9	1104	2	1	1	1	1
10	1104	-9	1	2	1,4142	1,4142
11	1104	1	1	3	1,7321	1,7321
12	1104	-9	1	4	2	2
13	1104	-9	1	5	2,2361	2,2361
14	1104	2	1	6	2,4495	2,4495
15	1105	2	1	0	0	0
16	1105	2	1	1	1	1
17	1105	-9	1	2	1,4142	1,4142
18	1105	1	1	3	1,7321	1,7321
19	1105	-9	1	4	2	2
20	1105	-9	1	5	2,2361	2,2361
21	1105	-9	1	6	2,4495	2,4495
22	1106	2	1	0	0	0
23	1106	1	1	1	1	1
24	1106	-9	1	2	1,4142	1,4142
25	1106	1	1	3	1,7321	1,7321
26	1106	-9	1	4	2	2
27	1106	-9	1	5	2,2361	2,2361
28	1106	1	1	6	2,4495	2,4495
29	1107	3	0	0	0	0
30	1107	3	0	1	1	0
31	1107	-9	0	2	1,4142	0
32	1107	3	0	3	1,7321	0
33	1107	-9	0	4	2	0
34	1107	-9	0	5	2,2361	0
35	1107	4	0	6	2,4495	0
36	1108	4	1	0	0	0
37	1108	4	1	1	1	1

38	1108	-9	1	2	1,4142	1,4142
39	1108	2	1	3	1,7321	1,7321
40	1108	-9	1	4	2	2
41	1108	-9	1	5	2,2361	2,2361
42	1108	3	1	6	2,4495	2,4495
43	1109	2	1	0	0	0
44	1109	1	1	1	1	1
45	1109	-9	1	2	1,4142	1,4142
46	1109	1	1	3	1,7321	1,7321
47	1109	-9	1	4	2	2
48	1109	-9	1	5	2,2361	2,2361
49	1109	2	1	6	2,4495	2,4495
50	1110	2	1	0	0	0
51	1110	3	1	1	1	1
52	1110	-9	1	2	1,4142	1,4142
53	1110	2	1	3	1,7321	1,7321
54	1110	-9	1	4	2	2
55	1110	-9	1	5	2,2361	2,2361
56	1110	2	1	6	2,4495	2,4495
57	1111	4	1	0	0	0
58	1111	4	1	1	1	1
59	1111	-9	1	2	1,4142	1,4142
60	1111	4	1	3	1,7321	1,7321
61	1111	-9	1	4	2	2
62	1111	-9	1	5	2,2361	2,2361
63	1111	2	1	6	2,4495	2,4495
64	1112	3	0	0	0	0
65	1112	-9	0	1	1	0
66	1112	-9	0	2	1,4142	0
67	1112	2	0	3	1,7321	0
68	1112	-9	0	4	2	0
69	1112	-9	0	5	2,2361	0
70	1112	2	0	6	2,4495	0
71	1113	2	1	0	0	0
72	1113	2	1	1	1	1
73	1113	-9	1	2	1,4142	1,4142
74	1113	3	1	3	1,7321	1,7321
75	1113	-9	1	4	2	2
76	1113	-9	1	5	2,2361	2,2361
77	1113	2	1	6	2,4495	2,4495
78	1114	4	1	0	0	0
79	1114	4	1	1	1	1
80	1114	-9	1	2	1,4142	1,4142
81	1114	4	1	3	1,7321	1,7321
82	1114	-9	1	4	2	2
83	1114	-9	1	5	2,2361	2,2361

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Output MIXOR 2.0 *for Windows* untuk Contoh 4.2

```
MIXOR - The program for mixed-effects ordinal regression analysis
          (version 2)

 NIMH Schiz Data - two grps - seven timepoints
 IMPS79 (ordinal) across SQRT Week - two random effects

 Response function: normal
 Random-effects distribution: normal

 Covariate(s) and random-effect(s) mean subtracted from thresholds
 ==> positive coefficient = positive association between regressor
     and ordinal outcome

 Numbers of observations
 -----------------------
 Level 1 observations =   1603
 Level 2 observations =    437

 The number of level 1 observations per level 2 unit are:

   4   4   3   4   4   4   4   4   4   3   4   4   4   2   3   4   3   4   3
   4   4   4   3   3   2   4   4   4   4   4   3   4   4   4   4   4   4   4
   4   4   2   3   4   3   4   4   4   3   4   4   2   2   4   5   4   2   4
   4   3   4   4   3   2   3   4   4   4   4   4   4   2   4   4   4   5   4
   4   2   2   4   2   4   4   3   3   4   4   4   4   4   4   4   4   3   3
   4   2   3   4   4   4   2   5   3   4   4   2   4   4   4   2   4   4   4
   4   4   4   4   4   4   5   2   4   3   4   4   2   2   4   4   4   4   4
   2   4   4   4   4   4   4   4   4   4   4   4   4   4   2   4   4   2   4
   4   4   3   4   2   4   4   3   2   3   4   4   3   3   4   3   4   4   4
   4   4   4   4   4   4   4   4   4   4   4   2   3   3   5   4   3   4   4
   3   2   4   4   4   4   4   3   3   4   4   4   4   4   4   4   4   4   4
   4   4   4   4   4   3   4   4   4   4   4   4   2   3   4   4   4   2   4
   4   4   4   3   4   4   4   4   4   4   4   4   3   4   4   3   4   4   2
   4   4   4   4   2   4   4   4   2   4   4   4   3   3   4   3   4   2   4
   4   4   3   3   4   4   4   4   3   3   4   3   4   4   4   4   4   3   4
   4   4   4   4   3   3   4   2   4   4   4   4   4   4   4   4   3   4   4
```

3	3	4	2	4	3	3	3	4	4	4	4	4	4	4	3	2	3	4
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	2	4	4	4	4	4	4	4	4	4	4	4	4	4
4	4	2	4	4	4	4	3	4	4	4	4	3	4	4	4	4	4	4
3	3	3	4	4	4	4	4	2	3	4	2	4	2	2	4	4	3	4
4	4	2	4	4	4	4	3	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	3	4	4	4	4	2	4	4	4	4	4	4

Descriptive statistics for all variables

Variable	Minimum	Maximum	Mean	Stand. Dev.
Imps79O	1.00000	4.00000	2.79601	1.02840
Intercpt	1.00000	1.00000	1.00000	0.00000
SqrtWeek	0.00000	2.44950	1.22041	0.89651
TxDrug	0.00000	1.00000	0.76419	0.42464
Tx*SWeek	0.00000	2.44950	0.94424	0.94541

Categories of the response variable Imps79O

Category	Frequency	Proportion
1.00	190.00	0.11853
2.00	474.00	0.29570
3.00	412.00	0.25702
4.00	527.00	0.32876

Crosstabulation of variable SqrtWeek by the response variable Imps79O

	Imps79O				
SqrtWeek	1.00	2.00	3.00	4.00	Total
0.00	1.0	54.0	122.0	257.0	434.0
	(0.00)	(0.12)	(0.28)	(0.59)	
1.00	23.0	135.0	124.0	144.0	426.0
	(0.05)	(0.32)	(0.29)	(0.34)	

```
    1.41     3.0     4.0     2.0     5.0    14.0
           (0.21)  (0.29)  (0.14)  (0.36)

    1.73    54.0   132.0   113.0    75.0   374.0
           (0.14)  (0.35)  (0.30)  (0.20)

    2.00     5.0     3.0     2.0     1.0    11.0
           (0.45)  (0.27)  (0.18)  (0.09)

    2.24     3.0     4.0     0.0     2.0     9.0
           (0.33)  (0.44)  (0.00)  (0.22)

    2.45   101.0   142.0    49.0    43.0   335.0
           (0.30)  (0.42)  (0.15)  (0.13)

Total      190.0   474.0   412.0   527.0  1603.0


Starting values
---------------
mean         0.835  0.264
covariates   0.028  0.345
var. terms   1.000  0.000  0.500
thresholds   1.037  1.700

==> The number of level 2 observations with non-varying responses
    =     79 (  18.08 percent )

---------------------------------------------------------
* Final Results - Maximum Marginal Likelihood Estimates *
---------------------------------------------------------
Total Iterations  =  21
Quad Pts per Dim  =  10
Log Likelihood    =   -1663.352
Deviance (-2logL) =    3326.704
Ridge             =       0.000

Variable        Estimate     Stand. Error            Z          p-value
--------     ------------    ------------    ------------    -----------
Intercpt         4.10997         0.25191        16.31505     0.00000  (2)
SqrtWeek        -0.50548         0.13048        -3.87407     0.00011  (2)
TxDrug           0.03823         0.22470         0.17013     0.86491  (2)
Tx*SWeek        -0.95018         0.14885        -6.38358     0.00000  (2)
```

```
Random effect variance & covariance terms (Cholesky of var-covariance
matrix)
 Intercpt        1.48627         0.14130        10.51880     0.00000  (1)
 covariance     -0.31473         0.08992        -3.49999     0.00047  (2)
 SqrtWeek        0.73025         0.06950        10.50782     0.00000  (1)

 Thresholds (for identification: threshold 1 = 0)
      2          2.18411         0.10992        19.86965     0.00000  (1)
      3          3.65352         0.14423        25.33155     0.00000  (1)

 note: (1) = 1-tailed p-value
       (2) = 2-tailed p-value

 Calculation of the random effects variance-covariance matrix
 ------------------------------------------------------------
 Intercpt varianãe = (1.486 * 1.486) = 2.209
        covariance = (1.486 * -.315) = -.468
 SqrtWeek variance = (-.315 * -.315) + (0.730 * 0.730) = 0.632

 Covariance expressed as a correlation = -.396
```