LAMPIRAN

LAMPIRAN

LAMPIRAN 1

Bukti Lemma 2.18.

Misalkan matriks $\mathbf{A}$ berukuran $n \times n$ merupakan matriks definit positif. Lemma 2.21 akan dibuktikan dengan menggunakan kontradiksi. Andaikan $\mathbf{A}$ merupakan matriks yang *singular* atau secara ekuivalen, yaitu $rank(\mathbf{A}) < n$. Akibatnya, kolom $\mathbf{A}$ tidak bebas linier sehingga terdapat vektor tidak nol $\mathbf{x}_*$, sedemikian sehingga $\mathbf{A}\,\mathbf{x}_* = \mathbf{0}$. Oleh karena itu,

$$\mathbf{x}_*^T \mathbf{A}\,\mathbf{x}_* = \mathbf{x}_*^T (\mathbf{A}\,\mathbf{x}_*) = \mathbf{x}_*^T \mathbf{0} = 0 .$$

Padahal $\mathbf{A}$ merupakan matriks definit positif, artinya

$$\mathbf{x}^\mathbf{T} \mathbf{A}\,\mathbf{x} > 0, \text{ untuk setiap } \mathbf{x} \neq 0 .$$

Sedangkan, jika $\mathbf{A}$ merupakan matriks *singular* diperoleh bahwa:

$$\mathbf{x}_*^T \mathbf{A}\,\mathbf{x}_* = \mathbf{x}_*^T (\mathbf{A}\,\mathbf{x}_*) = \mathbf{x}_*^T \mathbf{0} = 0$$

Berarti kontradiksi dengan pengandaian yang digunakan sehingga sembarang matriks definit positif adalah *nonsingular*.

LAMPIRAN 2

Bukti (2.3.2.a), (2.3.2.b), dan (2.3.2.c).

Misalkan a_i dinyatakan sebagai entri ke-i dari **a** dan x_i sebagai entri ke-i dari **x**, sedangkan a_{ik} dinyatakan sebagai entri ke-ik dari **A**. Maka

$$\mathbf{a}^T\mathbf{x} = \sum_i a_i x_i \text{ dan } \mathbf{x}^T\mathbf{A}\,\mathbf{x} = \sum_{i,k} a_{ik} x_i x_k$$

Sebagai langkah awal akan didefinisikan:

$$\frac{\partial x_i}{\partial x_j} = \begin{cases} 1, & \text{jika } i = j \\ 0, & \text{jika } i \neq j \end{cases}$$

$$\frac{\partial(x_i, x_k)}{\partial x_j} = \begin{cases} 2x_j, & \text{jika } i = k = j \\ x_i, & \text{jika } k = j \text{ tetapi } i \neq j \\ x_k, & \text{jika } i = j \text{ tetapi } k \neq j \\ 0, & \text{lainnya.} \end{cases}$$

sehingga diperoleh

$$\frac{\partial(\mathbf{a}^T\mathbf{x})}{\partial x_j} = \frac{\partial\left(\sum_i a_i x_i\right)}{\partial x_j} = \sum_i a_i \frac{\partial x_i}{\partial x_j} = a_j \qquad \textbf{(L2.1)}$$

dan

$$\frac{\partial(\mathbf{x}'\mathbf{A}\,\mathbf{x})}{\partial x_j} = \frac{\partial\left(\sum_{i,k} a_{ik} x_i x_k\right)}{\partial x_j}$$

$$= \frac{\partial\left(a_{jj}x_j^2 + \sum_{i\neq j} a_{ij}x_i x_j + \sum_{k\neq j} a_{jk}x_j x_k + \sum_{i\neq j,k\neq j} a_{ik}x_i x_k\right)}{\partial x_j}$$

$$= a_{jj}\frac{\partial x_j^2}{\partial x_j} + \sum_{i\neq j} a_{ij}\frac{\partial\left(x_i x_j\right)}{\partial x_j} + \sum_{k\neq j} a_{jk}\frac{\partial\left(x_j x_k\right)}{\partial x_j} + \sum_{i\neq j,k\neq j} a_{ik}\frac{\partial\left(x_i x_k\right)}{\partial x_j}$$

$$= 2a_{jj}x_j + \sum_{i\neq j} a_{ij}x_i + \sum_{k\neq j} a_{jk}x_k + 0$$

$$= \sum_i a_{ij}x_i + \sum_k a_{jk}x_k. \qquad \textbf{(L2.2)}$$

Persamaan (L2.1) dapat dibentuk dalam notasi vektor sebagai

$$\frac{\partial\left(\mathbf{a}^T\mathbf{x}\right)}{\partial\mathbf{x}} = \mathbf{a} \quad \text{atau} \quad \frac{\partial\left(\mathbf{a}^T\mathbf{x}\right)}{\partial\mathbf{x}^T} = \mathbf{a}^T$$

Sedangkan persamaan (L2.2) dapat dibentuk dalam notasi matriks sebagai:

$$\frac{\partial\left(\mathbf{x}^T\mathbf{A}\mathbf{x}\right)}{\partial\mathbf{x}} = \left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{x}$$

dengan $\sum_k a_{jk}x_k$ adalah entri ke-*j* dari vektor kolom $\mathbf{A}\mathbf{x}$ dan $\sum_i a_{ij}x_i$ adalah entri ke-*j* dari $\mathbf{A}^T\mathbf{x}$.

LAMPIRAN 3

Bukti (2.3.3.a).

Misalkan $\mathbf{L} = a\mathbf{F} + b\mathbf{G}$. Elemen ke-*is* dari **L** adalah sebagai berikut:

$$l_{is} = af_{is} + bg_{is}.$$

Fungsi f_{is} dan g_{is} *continuously differentiable* di $\mathbf{c}$ sehingga l_{is} juga *continuously differentiable* di $\mathbf{x} = \mathbf{c}$, yakni:

$$\frac{\partial l_{is}}{\partial x_j} = a\frac{\partial f_{is}}{\partial x_j} + b\frac{\partial g_{is}}{\partial x_j}.$$

Oleh karena $\partial l_{is}/\partial x_j$, $\partial f_{is}/\partial x_j$, dan $\partial g_{is}/\partial x_j$ masing-masing merupakan elemen ke-*is* dari $\partial \mathbf{L}/\partial x_j$, $\partial \mathbf{F}/\partial x_j$, dan $\partial \mathbf{G}/\partial x_j$, berarti **L** juga *continuously differentiable* di $\mathbf{x} = \mathbf{c}$, yaitu:

$$\frac{\partial \mathbf{L}}{\partial x_j} = a\frac{\partial \mathbf{F}}{\partial x_j} + b\frac{\partial \mathbf{G}}{\partial x_j}.$$

LAMPIRAN 4

Bukti (2.3.3.b).

Misalkan $\mathbf{H}=\mathbf{FG}$. Elemen ke-*it* dari $\mathbf{H}$ adalah sebagai berikut:

$$h_{it}=\sum_{s=1}^{q} f_{is} g_{st} \text{ .}$$

Fungsi f_{is} dan g_{st} *continuously differentiable* di $\mathbf{c}$ sehingga $f_{is}g_{st}$ juga *continuously differentiable* di $\mathbf{x}=\mathbf{c}$, yakni:

$$\frac{\partial f_{is} g_{st}}{\partial x_j}=f_{is}\frac{\partial g_{st}}{\partial x_j}+\frac{\partial f_{is}}{\partial x_j} g_{st} \text{ .}$$

Oleh karena itu, h_{it} *continuously differentiable* di $\mathbf{x}=\mathbf{c}$ atau dapat ditulis

$$\frac{\partial h_{it}}{\partial x_j}=\sum_{s=1}^{q}\frac{\partial\left(f_{is} g_{st}\right)}{\partial x_j}=\sum_{s=1}^{q} f_{is}\frac{\partial g_{st}}{\partial x_j}+\sum_{s=1}^{q}\frac{\partial f_{is}}{\partial x_j} g_{st} \text{ .}$$

Oleh karena $\sum_{s=1}^{q} f_{is}\left(\partial g_{st}/\partial x_j\right)$ dan $\sum_{s=1}^{q}\left(\partial f_{is}/\partial x_j\right) g_{st}$ masing-masing adalah elemen ke-*it* dari $\mathbf{F}\left(\partial\mathbf{G}/\partial x_j\right)$ dan $\left(\partial\mathbf{F}/\partial x_j\right)\mathbf{G}$), dapat disimpulkan bahwa **H** juga *continuously differentiable* di $\mathbf{x}=\mathbf{c}$, yaitu:

$$\frac{\partial\mathbf{FG}}{\partial x_j}=\mathbf{F}\frac{\partial\mathbf{G}}{\partial x_j}+\frac{\partial\mathbf{F}}{\partial x_j}\mathbf{G} \text{ .}$$

LAMPIRAN 5

Bukti (2.3.7.a), (2.3.7.b), dan (2.3.7.c).

Berdasarkan (2.3.3.b), (2.3.4.a), dan (2.3.6.c) didapatkan bahwa

$$\frac{\partial^2 \det(\mathbf{F})}{\partial x_i \partial x_j} = \frac{\partial\left\{|\mathbf{F}|\, tr\left(\mathbf{F}^{-1}\partial\mathbf{F}/\partial x_j\right)\right\}}{\partial x_i}$$

$$= |\mathbf{F}|\left[tr\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i \partial x_j}\right) + tr\left(-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\right)\right]$$

$$+|\mathbf{F}|\, tr\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\right) tr\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\right)$$

$$= |\mathbf{F}|\begin{bmatrix} tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_i \partial x_j}\right) + tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_i}\right) tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_j}\right) \\ -tr\left(\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\dfrac{\partial\mathbf{F}}{\partial x_j}\right) \end{bmatrix}$$

dan dengan menggunakan (2.3.3.c) dan (2.3.6.e) didapatkan pula bahwa

$$\frac{\partial^2 \mathbf{F}^{-1}}{\partial x_i \partial x_j} = \frac{\partial\left[-\mathbf{F}^{-1}\left(\partial\mathbf{F}/\partial x_j\right)\mathbf{F}^{-1}\right]}{\partial x_i}$$

$$= -\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\frac{\partial\mathbf{F}^{-1}}{\partial x_i} - \mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i \partial x_j}\mathbf{F}^{-1} - \frac{\partial\mathbf{F}^{-1}}{\partial x_i}\frac{\partial\mathbf{F}}{\partial x_j}\mathbf{F}^{-1}$$

$$= -\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\left(-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\right) - \mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i \partial x_j}\mathbf{F}^{-1}$$

$$-\left(-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\right)\frac{\partial\mathbf{F}}{\partial x_j}\mathbf{F}^{-1}$$

$$= -\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i \partial x_j}\mathbf{F}^{-1} + \mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\mathbf{F}^{-1}$$

$$+\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_j}\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial x_i}\mathbf{F}^{-1}.$$

Selanjutnya, berdasarkan (2.3.4.a), (2.3.6.d), dan (2.3.6.e) didapatkan bahwa

$$\frac{\partial^2 \log\det(\mathbf{F})}{\partial x_i \partial x_j} = \frac{\partial tr\left[\mathbf{F}^{-1}\left(\partial \mathbf{F}/\partial x_j\right)\right]}{\partial x_i}$$

$$= tr\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i \partial x_j}\right) + tr\left(-\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_j}\right)$$

$$= tr\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i \partial x_j}\right) - tr\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_i}\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x_j}\right).$$

LAMPIRAN 6

Bukti (2.4.b)

$(\rightarrow)$ Diketahui $T(\mathbf{y}) = c + \boldsymbol{l}^T\mathbf{y}$ dan $T(\mathbf{y})$ merupakan taksiran yang *unbiased* dan linier terhadap $\boldsymbol{\lambda}^T\boldsymbol{\alpha} + \boldsymbol{\omega}^T\boldsymbol{\beta}$ maka

$$\begin{aligned} E(T(\mathbf{y})) &= E(c + \boldsymbol{l}^T\mathbf{y}) \\ &= E(c) + E(\boldsymbol{l}^T(\mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\mathbf{b} + \mathbf{e})) \\ &= c + E(\boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha}) + E(\boldsymbol{l}^T\mathbf{Z}\mathbf{b}) + E(\boldsymbol{l}^T\mathbf{e}) \\ &= c + \boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha} + \boldsymbol{l}^T\mathbf{Z}E(\mathbf{b}) + \boldsymbol{l}^T E(\mathbf{e}) \\ &= c + \boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha} \qquad \text{[karena } E(\mathbf{b}) = \mathbf{0} \text{ dan } E(\mathbf{e}) = \mathbf{0}\text{]} \end{aligned} \qquad \textbf{(L6.1)}$$

sedangkan

$$\begin{aligned} E(\boldsymbol{\lambda}^T\boldsymbol{\alpha} + \boldsymbol{\omega}^T\mathbf{b}) &= E(\boldsymbol{\lambda}^T\boldsymbol{\alpha}) + E(\boldsymbol{\omega}^T\mathbf{b}) \\ &= \boldsymbol{\lambda}^T\boldsymbol{\alpha} + \boldsymbol{\omega}^T E(\mathbf{b}) \\ &= \boldsymbol{\lambda}^T\boldsymbol{\alpha} \qquad \text{[karena } E(\mathbf{b}) = \mathbf{0}\text{].} \end{aligned} \qquad \textbf{(L6.2)}$$

Oleh karena $T(\mathbf{y})$ *unbiased* terhadap $\boldsymbol{\lambda}^T\boldsymbol{\alpha} + \boldsymbol{\omega}^T\boldsymbol{\beta}$, terlihat bahwa persamaan (L6.1) akan sama dengan (L6.2) jika $c = 0$ dan $\boldsymbol{\lambda}^T = \boldsymbol{l}^T\mathbf{X}$.

[

$(\leftarrow)$ Diketahui bahwa $c = 0$ dan $\boldsymbol{\lambda}^T = \boldsymbol{l}^T\mathbf{X}$ maka

$$\begin{aligned} T(\mathbf{y}) &= c + \boldsymbol{l}^T\mathbf{y} \\ &= \boldsymbol{l}^T(\mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\mathbf{b} + \mathbf{e}) \\ &= \boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha} + \boldsymbol{l}^T\mathbf{Z}\mathbf{b} + \boldsymbol{l}^T\mathbf{e}. \end{aligned}$$

Berikut ini akan dibuktikan bahwa $T(\mathbf{y})$ dapat mengestimasi kombinasi linier

$\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$

$$
\begin{aligned}
E(T(\mathbf{y})) &= E\left(\boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha}+\boldsymbol{l}^T\mathbf{Zb}+\boldsymbol{l}^T\mathbf{e}\right)\\
&= E\left(\boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha}\right)+E\left(\boldsymbol{l}^T\mathbf{Zb}\right)+E\left(\boldsymbol{l}^T\mathbf{e}\right)\\
&= \boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha}+\boldsymbol{l}^T\mathbf{Z}E(\mathbf{b})+\boldsymbol{l}^T E(\mathbf{e})\\
&= \boldsymbol{l}^T\mathbf{X}\boldsymbol{\alpha} \quad \text{[karena } E(\mathbf{b})=\mathbf{0} \text{ dan } E(\mathbf{e})=\mathbf{0}\text{]}\\
&= \boldsymbol{\lambda}^T\boldsymbol{\alpha} \quad \text{[karena } \boldsymbol{\lambda}^T=\boldsymbol{l}^T\mathbf{X}\text{].}
\end{aligned}
$$

Jadi terbukti bahwa $T(\mathbf{y})$ unbiased terhadap kombinasi linier $\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$.

Selanjutnya akan dibuktikan bahwa $T(\mathbf{y})$ merupakan taksiran yang linier, yaitu

$T(a\mathbf{y}_1+b\mathbf{y}_2)=aT(\mathbf{y}_1)+bT(\mathbf{y}_2)$ dengan $T(\mathbf{y})=\boldsymbol{l}^T\mathbf{y}$.

$$
\begin{aligned}
T(a\mathbf{y}_1+b\mathbf{y}_2) &= \boldsymbol{l}^T(a\mathbf{y}_1+b\mathbf{y}_2)\\
&= \boldsymbol{l}^T a\mathbf{y}_1+\boldsymbol{l}^T b\mathbf{y}_2
\end{aligned}
$$

(L6.3)

$$
\begin{aligned}
aT(\mathbf{y}_1)+bT(\mathbf{y}_2) &= a\left(\boldsymbol{l}^T\mathbf{y}_1\right)+b\left(\boldsymbol{l}^T\mathbf{y}_2\right)\\
&= a\boldsymbol{l}^T\mathbf{y}_1+b\boldsymbol{l}^T\mathbf{y}_2
\end{aligned}
$$

(L6.4)

Terlihat dari (L6.3) dan (L6.4) bahwa $T(\mathbf{y})$ merupakan taksiran yang linier. Oleh karena $T(\mathbf{y})$ merupakan taksiran yang linier dan *unbiased* terhadap kombinasi linier $\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$, maka terbukti bahwa $\boldsymbol{\lambda}^T\boldsymbol{\alpha}+\boldsymbol{\omega}^T\boldsymbol{\beta}$ dapat diestimasi oleh $T(\mathbf{y})$.

LAMPIRAN 7

Bukti (2.4.g)

$$E(\mathbf{y}) = E(\mathbf{X\alpha} + \mathbf{Zb} + \mathbf{e})$$
$$= E(\mathbf{X\alpha}) + E(\mathbf{Zb}) + E(\mathbf{e})$$
$$= \mathbf{X\alpha} + \mathbf{Z}E(\mathbf{b}) + E(\mathbf{e})$$
$$= \mathbf{X\alpha} \qquad \text{[karena } E(\mathbf{b}) = \mathbf{0} \text{ dan } E(\mathbf{e}) = \mathbf{0}\text{]}$$

$$Var(\mathbf{y}) = E\left((\mathbf{y} - \mathbf{X\alpha})(\mathbf{y} - \mathbf{X\alpha})^T\right)$$
$$= E\left((\mathbf{Zb} + \mathbf{e})(\mathbf{Zb} + \mathbf{e})^T\right)$$
$$= E\left((\mathbf{Zb} + \mathbf{e})(\mathbf{b}^T\mathbf{Z}^T + \mathbf{e}^T)\right)$$
$$= E\left(\mathbf{Zbb}^T\mathbf{Z}^T + \mathbf{Zbe}^T + \mathbf{eb}^T\mathbf{Z}^T + \mathbf{ee}^T\right)$$
$$= E\left(\mathbf{Zbb}^T\mathbf{Z}^T\right) + E\left(\mathbf{Zbe}^T\right) + E\left(\mathbf{eb}^T\mathbf{Z}^T\right) + E\left(\mathbf{ee}^T\right)$$
$$= \mathbf{Z}E\left(\mathbf{bb}^T\right)\mathbf{Z}^T + \mathbf{Z}E\left(\mathbf{be}^T\right) + E\left(\mathbf{eb}^T\right)\mathbf{Z}^T + \mathbf{R}$$
$$= \mathbf{ZGZ}^T + \mathbf{R} \qquad \text{[karena } E\left(\mathbf{be}^T\right) = E\left(\mathbf{eb}^T\right) = \mathbf{0}\text{]}$$
$$= \mathbf{\Omega} \qquad \text{[karena } \mathbf{\Omega} = \mathbf{R} + \mathbf{ZGZ}^T.\text{]}$$

Oleh karena **b** dan **e** berdistribusi normal, maka **y** pun berdistribusi normal atau ditulis sebagai $\mathbf{y} \sim N(\mathbf{X\alpha}, \mathbf{\Omega})$ di mana pdf-nya adalah sebagai berikut:

$$f(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}\left|\mathbf{\Omega}(\boldsymbol{\delta})\right|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mathbf{X\alpha})^T \mathbf{\Omega}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X\alpha})\right)$$

LAMPIRAN 8

Bukti (2.4.i).

$$s_j(\boldsymbol{\alpha},\boldsymbol{\delta}) = \frac{\partial \ln L(\boldsymbol{\alpha},\boldsymbol{\delta};\mathbf{y})}{\partial \delta_j} = \frac{\partial\left(c - \frac{1}{2}\left(\ln\left(\left|\boldsymbol{\Omega}(\boldsymbol{\delta})\right|\right) + (\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T \boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)\right)}{\partial \delta_j}$$

$$= \frac{\partial c}{\partial \delta_j} - \frac{1}{2}\frac{\partial \ln\left(\left|\boldsymbol{\Omega}(\boldsymbol{\delta})\right|\right)}{\partial \delta_j} - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T \frac{\partial \boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})}{\partial \delta_j}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})$$

$$= -\frac{1}{2} tr\left(\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\frac{\partial \boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial \delta_j}\right) + \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\left(\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\frac{\partial \boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial \delta_j}\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})$$

$$= -\frac{1}{2} tr\left(\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\frac{\partial \boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial \delta_j}\right) + \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\left(\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\frac{\partial \boldsymbol{\Omega}(\boldsymbol{\delta})}{\partial \delta_j}\boldsymbol{\Omega}^{-1}(\boldsymbol{\delta})\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})$$

LAMPIRAN 9

Bukti (2.4.k)

Dengan menggunakan (2.4.3), (2.16.2), dan (2.16.3) akan dicari $\boldsymbol{\Upsilon}(\boldsymbol{\delta})$ sehingga *Scoring Algorithm* dapat digunakan. Misalkan $\Upsilon_{jk}(\boldsymbol{\delta})$ adalah elemen ke-*jk* dari $\boldsymbol{\Upsilon}(\boldsymbol{\delta})$,

yaitu:

$$\Upsilon_{jk}(\boldsymbol{\delta}) = -E\left(\frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\delta};\mathbf{y})}{\partial\delta_j\partial\delta_k}\right)$$

$$= -E\left(\frac{\partial^2\left(c - \frac{1}{2}\left(\ln\left(|\boldsymbol{\Omega}|\right) + (\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\boldsymbol{\Omega}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)\right)}{\partial\delta_j\partial\delta_k}\right)$$

$$= -E\left(\frac{\partial^2 c}{\partial\delta_j\partial\delta_k} - \frac{1}{2}\frac{\partial^2\ln\left(|\boldsymbol{\Omega}|\right)}{\partial\delta_j\partial\delta_k} - \frac{1}{2}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T\frac{\partial^2\boldsymbol{\Omega}^{-1}}{\partial\delta_j\partial\delta_k}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})\right)$$

$$= -E\begin{pmatrix} -\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\frac{\partial^2\boldsymbol{\Omega}}{\partial\delta_j\partial\delta_k}\right) - tr\left(\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_k}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_j}\right)\right) - \frac{1}{2}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T \\ \left(-\boldsymbol{\Omega}^{-1}\frac{\partial^2\boldsymbol{\Omega}}{\partial\delta_j\partial\delta_k}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_k}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_j}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_j}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\delta_k}\boldsymbol{\Omega}^{-1}\right)(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}}) \end{pmatrix}$$

$$= -E\begin{pmatrix} -\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\right) - tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\right)\right) - \frac{1}{2}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T \\ \left(-\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\right)(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}}) \end{pmatrix}$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\right) - tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\right)\right) - \frac{1}{2}tr\left((\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T\begin{pmatrix} -\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1} \end{pmatrix}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})\right)\right)$$

$$\Upsilon_{jk}(\boldsymbol{\delta}) = -E\left(-\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\right) - tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\right)\right) - \frac{1}{2}tr\left(\begin{pmatrix} -\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1} \end{pmatrix}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T\right)\right)$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\right) - tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\right) + tr\left(\begin{pmatrix} -\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1} \end{pmatrix}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T\right)\right)\right)$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)} + \begin{pmatrix} -\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1} \end{pmatrix}(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T\right)\right)\right)$$

$$= \frac{1}{2}tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)} + \begin{pmatrix} -\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} \\ +\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1} \end{pmatrix} E\left((\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})(\mathbf{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}})^T\right)\right)$$

$$= \frac{1}{2}tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)} + \left(-\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\right)\boldsymbol{\Omega}\right)$$

$$= \frac{1}{2}tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)} + \left(-\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\right)\right)$$

$$= \frac{1}{2}tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j,k)} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)} + \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\right)$$

$$= \frac{1}{2}tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\right).$$

LAMPIRAN 10

Menunjukkan $\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t$ dan $\mathbf{d}^T\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha}\right)$ tidak berkorelasi

Bukti:

$\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t$ dan $\mathbf{d}^T\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha}\right)$ tidak berkorelasi jika

$\operatorname{cov}\left[\left(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t\right),\mathbf{d}^T\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha}\right)\right]=0$.

$$\operatorname{cov}\left[\left(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t\right),\mathbf{d}^T\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha}\right)\right]=E\left[\left(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t\right)-E\left(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t\right)\right]$$
$$\left[\mathbf{d}^T\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha}\right)-E\left(\mathbf{d}^T\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha}\right)\right)\right]^T$$

Pertama akan dicari:

1. $$\begin{aligned}
\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta}) &= \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{y}\\
\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta}) &= \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\\
\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta}) &= \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\boldsymbol{\alpha}+\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\\
\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta}) &= \boldsymbol{\alpha}+\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\\
\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha} &= \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)
\end{aligned}$$

2. $$\begin{aligned}
E\left[\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-\mathbf{t}\right] &= E\left[\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\alpha}+\boldsymbol{\omega}^{\mathbf{T}}\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}\right)-\left(\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\alpha}+\boldsymbol{\omega}^{\mathbf{T}}\mathbf{b}\right)\right]\\
&= E\left[\boldsymbol{\omega}^{\mathbf{T}}\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)-\boldsymbol{\omega}^{\mathbf{T}}\mathbf{b}\right]\\
&= E\left[\boldsymbol{\omega}^{\mathbf{T}}\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{b}+\boldsymbol{\omega}^{\mathbf{T}}\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{e}-\boldsymbol{\omega}^{\mathbf{T}}\mathbf{b}\right]\\
&= \boldsymbol{\omega}^{\mathbf{T}}\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}E\left[\mathbf{b}\right]+\boldsymbol{\omega}^{\mathbf{T}}\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}E\left[\mathbf{e}\right]-\boldsymbol{\omega}^{\mathbf{T}}E\left[\mathbf{b}\right]\\
&= 0
\end{aligned}$$

3. $E\left[\mathbf{d}^{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\right]=E\left[\boldsymbol{\lambda}^{\mathrm{T}}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\right]$

$$=E\left[\boldsymbol{\lambda}^{\mathrm{T}}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\right)\right]$$

$$=E\left[\boldsymbol{\lambda}^{\mathrm{T}}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{b}+\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{e}\right)\right]$$

$$=E\left[\left(\boldsymbol{\lambda}^{\mathrm{T}}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\right)\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{b}+\left(\boldsymbol{\lambda}^{\mathrm{T}}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\right)\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{e}\right]$$

$$=\left(\boldsymbol{\lambda}^{\mathrm{T}}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\right)\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}E\left[\mathbf{b}\right]+\left(\boldsymbol{\lambda}^{\mathrm{T}}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\right)\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}E\left[\mathbf{e}\right]$$

$$=0$$

Berdasarkan (1), (2), dan (3) maka $\mathrm{cov}\left[\left(\hat{\boldsymbol{\tau}}^{*}\left(\delta,\boldsymbol{\alpha},\mathbf{y}\right)-t\right),\mathbf{d}^{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\right]$ menjadi:

$$\mathrm{cov}\left[\left(\hat{\boldsymbol{\tau}}^{*}\left(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y}\right)-t\right),\mathbf{d}^{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\right]=E\left[\left(\hat{\boldsymbol{\tau}}^{*}\left(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y}\right)-t\right)-E\left(\hat{\boldsymbol{\tau}}^{*}\left(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y}\right)-t\right)\right]$$

$$\left[\mathbf{d}^{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)-E\left(\mathbf{d}^{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\right)\right]^{T}$$

$$=E\left[\left(\hat{\boldsymbol{\tau}}^{*}\left(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y}\right)-t\right)\right]\left[\mathbf{d}^{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\right]^{T}$$

$$=E\left[\boldsymbol{\lambda}^{T}\boldsymbol{\alpha}+\boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{T}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{x}\boldsymbol{\alpha}\right)-\left(\boldsymbol{\lambda}^{T}\boldsymbol{\alpha}+\boldsymbol{\omega}^{T}\mathbf{b}\right)\right]$$

$$\left[\left(\boldsymbol{\lambda}^{T}-\boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{T}\boldsymbol{\Omega}^{-1}\right)\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\right]^{T}$$

$$
\begin{aligned}
&= E\left[\boldsymbol{\omega}^T \mathbf{G}\mathbf{Z}^T \boldsymbol{\Omega}^{-1}(\mathbf{y}-\mathbf{x}\boldsymbol{\alpha}) - \boldsymbol{\omega}^T\mathbf{b}\right] \\
&\quad \left[\left(\boldsymbol{\lambda}^T - \boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\right)\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right]^T \\
&= E\left[\boldsymbol{\omega}^T \mathbf{G}\mathbf{Z}^T \boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e}) - \boldsymbol{\omega}^T\mathbf{b}\right] \\
&\quad \left[(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\right)^T\left(\boldsymbol{\lambda}^T - \boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\right)^T\right] \\
&= E\left[\left(\mathbf{a}^T(\mathbf{Zb}+\mathbf{e}) - \boldsymbol{\omega}^T\mathbf{b}\right)(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\right] \\
&\quad \left[\mathbf{X}\left(\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\right)^T\left(\boldsymbol{\lambda}^T - \boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\right)^T\right] \\
&= 0
\end{aligned}
$$

(L11.1)

Karena $\operatorname{cov}\left[\left(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t\right),\mathbf{d}^T(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})\right]=0$ maka $\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\alpha},\mathbf{y})-t$ dan $\mathbf{d}^T(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})$

tidak berkorelasi, sehingga:

$$
\begin{aligned}
MSE\left[\hat{\tau}(\delta,\mathrm{y})\right] &= E\left[\hat{\tau}(\delta,\mathrm{y})-t\right]^2 \\
&= E\left[\hat{\tau}^*(\delta,\alpha,\mathrm{y})+d^T(\hat{\alpha}-\alpha)-t\right]^2 \\
&= E\left[\left(\hat{\tau}^*(\delta,\alpha,\mathrm{y})-t\right)+d^T(\hat{\alpha}-\alpha)\right]^2 \\
&= E\left[\hat{\tau}^*(\delta,\alpha,\mathrm{y})-t\right]^2+E\left[d^T(\hat{\alpha}-\alpha)\right]^2+2\operatorname{cov}\left[\left(\hat{\tau}^*(\delta,\alpha,\mathrm{y})-t\right),d^T(\hat{\alpha}-\alpha)\right] \\
&= E\left[\hat{\tau}^*(\delta,\alpha,\mathrm{y})-t\right]^2+E\left[d^T(\hat{\alpha}-\alpha)\right]^2
\end{aligned}
$$

(L11.2)

LAMPIRAN 11

Menunjukkan $\hat{\tau}^*(\delta,\alpha,\mathrm{y})$ unbiased terhadap t dan $\hat{\alpha}$ unbiased terhadap α

1. Akan dibuktikan bahwa $\hat{\tau}^*(\delta,\alpha,\mathrm{y})$ unbiased terhadap t.

   Bukti:

$$\begin{aligned}
E\left[\hat{\tau}^*(\delta,\alpha,\mathbf{y})-\mathbf{t}\right] &= E\left[\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\alpha}+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\alpha}+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right)\right] \\
&= E\left[\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Z}\mathbf{b}+\mathbf{e})-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right] \\
&= E\left[\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{b}+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{e}-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right] \\
&= \boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}E[\mathbf{b}]+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}E[\mathbf{e}]-\boldsymbol{\omega}^{\mathrm{T}}E[\mathbf{b}] \\
&= 0
\end{aligned}$$

   $\therefore$ terbukti bahwa $\hat{\tau}^*(\delta,\alpha,\mathrm{y})$ unbiased terhadap t.

2. Akan dibuktikan bahwa $\hat{\alpha}$ unbiased terhadap α.

3. Bukti

$$\begin{aligned}
E\left[\hat{\alpha}(\delta)\right] &= E\left[\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{y}\right] \\
&= E\left[\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Z}\mathbf{b}+\mathbf{e})\right] \\
&= E\left[\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\boldsymbol{\alpha}+\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Z}\mathbf{b}+\mathbf{e})\right], \quad E[\mathbf{b}]=0, E[\mathbf{e}]=0 \\
&= \boldsymbol{\alpha}
\end{aligned}$$

   $\therefore$ terbukti bahwa $\hat{\alpha}$ unbiased terhadap α.

LAMPIRAN 12

Akan ditunjukkan bentuk $g_1(\boldsymbol{\delta}) = \boldsymbol{\lambda}^{\mathrm{T}}\left(\mathbf{G} - \mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}\right)\boldsymbol{\lambda}$ dan

$g_2(\boldsymbol{\delta}) = \mathbf{d}^{\mathrm{T}}\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d}$

$$
\begin{aligned}
g_1(\boldsymbol{\delta}) &= \operatorname{var}\left[\hat{\tau}^*(\delta, \alpha, \mathrm{y}) - \mathbf{t}\right] \\
&= E\left[\left(\lambda^T\alpha + \mathbf{a}^T(\mathbf{y} - \mathbf{X}\alpha) - \left(\lambda^T\alpha + \omega^T\mathbf{b}\right)\right)\left(\lambda^T\alpha + \mathbf{a}^T(\mathbf{y} - \mathbf{X}\alpha) - \left(\lambda^T\alpha + \omega^T\mathbf{b}\right)\right)^T\right] \\
&= E\left[\left(\mathbf{a}^T(\mathbf{y} - \mathbf{X}\alpha) - \omega^T\mathbf{b}\right)\left(\mathbf{a}^T(\mathbf{y} - \mathbf{X}\alpha) - \omega^T\mathbf{b}\right)^T\right] \\
&= E\left[\left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)\left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)^T\right] \\
&= E\left[\left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)\left((\mathbf{Zb} + \mathbf{e})^T\mathbf{a} - \mathbf{b}^T\omega\right)\right] \\
&= E\left[\left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)(\mathbf{Zb} + \mathbf{e})^T\mathbf{a} - \left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)\mathbf{b}^T\omega\right] \\
&= E\left[\left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)(\mathbf{Zb} + \mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\omega - \left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)\mathbf{b}^T\omega\right] \\
&= E\left[\left(\left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)(\mathbf{Zb} + \mathbf{e})^T\boldsymbol{\Omega}^{-1}\right)\mathbf{ZG}^T\omega - \left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)\mathbf{b}^T\omega\right] \\
&= 0 - E\left[\left(\mathbf{a}^T(\mathbf{Zb} + \mathbf{e}) - \omega^T\mathbf{b}\right)\mathbf{b}^T\omega\right] \\
&= E\left[\left(\omega^T\mathbf{b} - \mathbf{a}^T(\mathbf{Zb} + \mathbf{e})\right)\mathbf{b}^T\omega\right] \\
&= E\left[\left(\omega^T\mathbf{b} - \omega^T\mathbf{GZ}^T\boldsymbol{\Omega}^{-1}(\mathbf{Zb} + \mathbf{e})\right)\mathbf{b}^T\omega\right] \\
&= E\left[\omega^T\left(\mathbf{b} - \mathbf{GZ}^T\boldsymbol{\Omega}^{-1}(\mathbf{Zb} + \mathbf{e})\right)\mathbf{b}^T\omega\right] \\
&= \omega^T E\left[\left(\mathbf{bb}^T - \mathbf{GZ}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{Zbb}^T + \mathbf{eb}^T\right)\right)\right]\omega \\
&= \omega^T\left(\mathbf{G} - \mathbf{GZ}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}\right)\omega
\end{aligned}
$$

$$
\begin{aligned}
g_2(\boldsymbol{\delta}) &= \operatorname{var}\left[\mathbf{d}^T(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})\right] \\
&= E\left[\mathbf{d}^T(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}).\left(\mathbf{d}^T(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})\right)^T\right] \\
&= E\left[\mathbf{d}^T(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})^T\mathbf{d}\right] \\
&= E\left[\mathbf{d}^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d}\right] \\
&= \mathbf{d}^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\right]\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d} \\
&= \mathbf{d}^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d} \\
&= \mathbf{d}^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{I}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d} \\
&= \mathbf{d}^T\left(\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d} \\
&= \mathbf{d}^T\mathbf{I}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d} \\
&= \mathbf{d}^T\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{d}
\end{aligned}
$$

Jadi, $MSE\left[\hat{\tau}(\boldsymbol{\delta},\mathrm{y})\right] = g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta})$

$$
= \boldsymbol{\omega}^T\left(\mathbf{G}-\mathbf{G}\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}\right)\boldsymbol{\omega} + \mathbf{d}^{\mathbf{T}}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)\mathbf{d}
$$

dengan

$$
\mathbf{d} = \boldsymbol{\lambda}^T - \boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\mathbf{X}
$$

LAMPIRAN 13

Menunjukkan bahwa dengan diberikan $\hat{\delta}$ *translation invariant*, serta diasumsikan **b** dan **e** berdistribusi normal, maka

$\left[\hat{\boldsymbol{\tau}}(\delta)-t\right]$ dan $\left[\hat{\boldsymbol{\tau}}\left(\hat{\delta}\right)-\hat{\boldsymbol{\tau}}(\delta)\right]$ independent.

Diketahui bahwa distribusi bersama dari $\hat{\tau}(\boldsymbol{\delta})-\boldsymbol{t}$ dan $\boldsymbol{y}-\mathbf{X}\hat{\boldsymbol{\alpha}}$ adalah multivariat normal, sehingga $\hat{\tau}(\boldsymbol{\delta})-\boldsymbol{t}$ independen terhadap $\boldsymbol{y}-\mathbf{X}\hat{\boldsymbol{\alpha}}$, dan setiap fungsi dari $\boldsymbol{y}-\mathbf{X}\hat{\boldsymbol{\alpha}}$ (Kackar, 1984).

Kemudian dengan diberikan bahwa $\hat{\delta}$ translation invariant dapat diperoleh $\hat{\tau}(\boldsymbol{\delta})-\boldsymbol{t}$ independen dengan $\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}(\boldsymbol{\delta})$.

Bukti:

Dengan asumsi $\hat{\delta}$ translation invariant didapatkan bahwa $\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}(\boldsymbol{\delta})$ bergantung pada **y** yaitu hanya pada $\boldsymbol{y}-\mathbf{X}\tilde{\boldsymbol{\alpha}}$, yang dapat dicari langsung sebagai berikut:

$$\begin{aligned}\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}(\boldsymbol{\delta}) &= \left(\boldsymbol{\lambda}^T\hat{\boldsymbol{\alpha}}\left(\hat{\boldsymbol{\delta}}\right)+\boldsymbol{\omega}^T\hat{\boldsymbol{\beta}}\left(\hat{\boldsymbol{\delta}}\right)\right)-\left(\boldsymbol{\lambda}^T\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})+\boldsymbol{\omega}^T\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\\ &= \boldsymbol{\lambda}^T\left(\hat{\boldsymbol{\alpha}}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})\right)+\boldsymbol{\omega}^T\left(\hat{\boldsymbol{\beta}}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\end{aligned}$$

(L14.1)

i. $\hat{\boldsymbol{\alpha}}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})=\left(\mathbf{X}^{\mathbf{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y}-\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{y}$

$$= \left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{y} + \hat{\alpha}(\delta) - \hat{\alpha}(\delta)$$

$$= \left[\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \hat{\alpha}(\delta)\right] - \left[\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{y} - \hat{\alpha}(\delta)\right]$$

$$= \left[\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\hat{\alpha}(\delta)\right] -$$

$$\left[\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{y} - \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\hat{\alpha}(\delta)\right]$$

$$= \left[\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)\right] - \left[\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)\right]$$

$$= \left[\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1} - \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\right]\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$$

ii. $\hat{\beta}\left(\hat{\delta}\right) - \hat{\beta}(\delta) = \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}\left(\hat{\delta}\right)\right) - \mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$

$$= \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\hat{\alpha}\left(\hat{\delta}\right) - \mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$$

$$= \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$$

$$= \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right) +$$

$$\hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\hat{\alpha}(\delta) - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\hat{\alpha}(\delta)$$

$$= \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\hat{\alpha}(\delta) - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} + \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\hat{\alpha}(\delta) -$$

$$\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$$

$$= \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\tilde{\alpha} - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y} + \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\tilde{\alpha} -$$

$$\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$$

$$= \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right) - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right) - \mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$$

$$= \left[\hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1} - \hat{\mathbf{G}}\mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\Omega}}^{-1} - \mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\right]\left(\mathbf{y} - \mathbf{X}\hat{\alpha}(\delta)\right)$$

(i)dan (ii) membuktikan bahwa persamaan (1) bergantung pada **y** yaitu hanya pada $y - \mathbf{X}\hat{\alpha}(\delta)$ sehingga persamaan **(L14.1)** merupakan fungsi yang translation invariant. Ambil :

a. $y - \mathbf{X}\hat{\alpha}(\delta)$ adalah maximal invariant, dan

b. Setiap fungsi yang translation invariant dari **y** dapat diekspresikan sebagai fungsi maximal invariant.

Sehingga, untuk membuktikan $\hat{\tau}(\delta) - t$ berdistribusi independent dengan $\hat{\tau}(\hat{\delta}) - \hat{\tau}(\delta)$ perlu dibuktikan $\text{cov}\left[\left(\hat{\tau}(\hat{\delta}) - \hat{\tau}(\delta)\right), \left(\hat{\tau}(\delta) - t\right)\right] = 0$.

Karena kondisi (a) dan (b) maka:

$$\begin{aligned}\text{cov}\left[\left(\hat{\tau}(\hat{\delta}) - \hat{\tau}(\delta)\right), \left(\hat{\tau}(\delta) - t\right)\right] &= \text{cov}\left[\left(y - \mathbf{X}\hat{\alpha}(\delta)\right), \left(\hat{\tau}(\delta) - t\right)\right] \\ &= E\left[\left(y - \mathbf{X}\hat{\alpha}(\delta)\right) - E\left[y - \mathbf{X}\hat{\alpha}(\delta)\right]\right]\left[\left(\hat{\tau}(\delta) - t\right) - E\left[\hat{\tau}(\delta) - t\right]\right]^T\end{aligned}$$

(L14.2)

$$\begin{aligned}E\left[y - \mathbf{X}\hat{\alpha}(\delta)\right] &= E\left[\mathbf{X}\alpha + \mathbf{Zb} + \mathbf{e} - \mathbf{X}\hat{\alpha}(\delta)\right] \\ &= E\left[\mathbf{X}\left(\alpha - \hat{\alpha}(\delta)\right) + \mathbf{Zb} + \mathbf{e}\right] \\ &= E\left[\mathbf{X}.-\left(\mathbf{X}^{\mathrm{T}}\Omega^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\Omega^{-1}\left(\mathbf{Zb} + \mathbf{e}\right) + \mathbf{Zb} + \mathbf{e}\right] \\ &= -\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\Omega^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\Omega^{-1}E\left[\left(\mathbf{Zb} + \mathbf{e}\right)\right] + \mathbf{Z}E[\mathbf{b}] + E[\mathbf{e}] \\ &= -\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\Omega^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\Omega^{-1}.0 + \mathbf{Z}.0 + 0 \\ &= 0\end{aligned} \qquad \text{(iii)}$$

$$E\left[\hat{\tau}(\delta)-t\right]=E\left[\left(\boldsymbol{\lambda}^{\mathrm{T}}\hat{\alpha}(\delta)+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)\right)-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\alpha}+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right)\right]$$

$$=E\left[\boldsymbol{\lambda}^{\mathrm{T}}\left(\hat{\alpha}(\delta)-\alpha\right)+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Z}\mathbf{b}+\mathbf{e}-\mathbf{X}\hat{\alpha}(\delta)\right)-\left(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right)\right]$$

$$=E\left[\boldsymbol{\lambda}^{\mathrm{T}}\left(\hat{\alpha}(\delta)-\alpha\right)+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{X}\left(\alpha-\hat{\alpha}(\delta)\right)+\mathbf{Z}\mathbf{b}+\mathbf{e}\right)-\left(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right)\right]$$

$$=E\Big[\boldsymbol{\lambda}^{\mathrm{T}}\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}$$

$$\left(\mathbf{X}.-\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)+\mathbf{Z}\mathbf{b}+\mathbf{e}\right)-\left(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right)\Big]$$

$$=\boldsymbol{\lambda}^{\mathrm{T}}\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}E\left[\mathbf{Z}\mathbf{b}+\mathbf{e}\right]-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}$$

$$\mathbf{X}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}E\left[\mathbf{Z}\mathbf{b}+\mathbf{e}\right]+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}E\left[\mathbf{Z}\mathbf{b}+\mathbf{e}\right]-\boldsymbol{\omega}^{\mathrm{T}}E\left[-(\mathbf{b})\right]$$

$$=0$$

(iv)

Dari (iii) dan (iv) maka **(L14.2)** menjadi:

$$\operatorname{cov}\left[\left(\hat{\tau}(\hat{\delta})-\hat{\tau}(\delta)\right),\left(\hat{\tau}(\delta)-t\right)\right]=E\left[\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)\right]\left[\left(\hat{\tau}(\delta)-t\right)\right]^{\mathrm{T}}$$

$$=E\left[\left(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Z}\mathbf{b}+\mathbf{e}\right)-\mathbf{X}\hat{\alpha}(\delta)\right]$$

$$\left[\left(\boldsymbol{\lambda}^{\mathrm{T}}\hat{\alpha}(\delta)+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)\right)-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\alpha}+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right)\right]^{\mathrm{T}}$$

$$=E\left[\mathbf{X}\left(\alpha-\hat{\alpha}(\delta)\right)+\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\right]$$

$$\left[\boldsymbol{\lambda}^{\mathrm{T}}\left(\hat{\alpha}(\delta)-\alpha\right)+\boldsymbol{\omega}^{\mathrm{T}}\left(\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)-\boldsymbol{\omega}^{\mathrm{T}}\mathbf{b}\right)\right]^{\mathrm{T}}$$

$$=E\left[\mathbf{X}\left(\alpha-\hat{\alpha}(\delta)\right)+\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\right]$$

$$\left[\left(\tilde{\alpha}-\alpha\right)^{\mathrm{T}}\boldsymbol{\lambda}+\left(\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)-\mathbf{b}\right)^{\mathrm{T}}\boldsymbol{\omega}\right]$$

$$=E\Big[\mathbf{X}\left(\alpha-\hat{\alpha}(\delta)\right)\left(\hat{\alpha}(\delta)-\alpha\right)^{\mathrm{T}}\boldsymbol{\lambda}+\mathbf{X}\left(\alpha-\hat{\alpha}(\delta)\right)\left(\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)-\mathbf{b}\right)^{\mathrm{T}}\boldsymbol{\omega}$$

$$\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\left(\hat{\alpha}(\delta)-\alpha\right)^{\mathrm{T}}\boldsymbol{\lambda}+\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\left(\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)-\mathbf{b}\right)^{\mathrm{T}}\boldsymbol{\omega}\Big]$$

$$=\underbrace{E\left[\mathbf{X}\left(\alpha-\hat{\alpha}(\delta)\right)\left(\hat{\alpha}(\delta)-\alpha\right)^{\mathrm{T}}\boldsymbol{\lambda}\right]}_{\mathrm{I}}+\underbrace{E\left[\mathbf{X}\left(\alpha-\hat{\alpha}(\delta)\right)\left(\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\tilde{\alpha}\right)-\mathbf{b}\right)^{\mathrm{T}}\boldsymbol{\omega}\right]}_{\mathrm{II}}+$$

$$\underbrace{E\left[\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\left(\hat{\alpha}(\delta)-\alpha\right)^{\mathrm{T}}\boldsymbol{\lambda}\right]}_{\mathrm{III}}+\underbrace{E\left[\left(\mathbf{Z}\mathbf{b}+\mathbf{e}\right)\left(\mathbf{G}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)-\mathbf{b}\right)^{\mathrm{T}}\boldsymbol{\omega}\right]}_{\mathrm{IV}}$$

I. $E\left[\mathbf{X}\left(\boldsymbol{\alpha}-\hat{\alpha}(\delta)\right)\left(\hat{\alpha}(\delta)-\boldsymbol{\alpha}\right)^{T}\boldsymbol{\lambda}\right]=\mathbf{X}E\left[\left(\boldsymbol{\alpha}-\hat{\alpha}(\delta)\right)\left(\hat{\alpha}(\delta)-\boldsymbol{\alpha}\right)^{T}\right]\boldsymbol{\lambda}$

$$=\mathbf{X}E\left[-\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\left(\mathbf{Zb}+\mathbf{e}\right)\left(\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\left(\mathbf{Zb}+\mathbf{e}\right)\right)^{T}\right]\boldsymbol{\lambda}$$

$$=\mathbf{X}E\left[-\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\left(\mathbf{Zb}+\mathbf{e}\right)\left(\mathbf{Zb}+\mathbf{e}\right)^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\right]\boldsymbol{\lambda}$$

$$=-\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}E\left[\left(\mathbf{Zb}+\mathbf{e}\right)\left(\mathbf{Zb}+\mathbf{e}\right)^{T}\right]\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

$$=-\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

$$=-\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{I}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

$$=-\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

II. $E\left[\mathbf{X}\left(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}(\delta)\right)\left(\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\alpha}}(\delta)\right)-\mathbf{b}\right)^{\mathbf{T}}\boldsymbol{\omega}\right]=E\left[\mathbf{X}.-\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right.$

$$\left.\left(\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Zb}+\mathbf{e}-\mathbf{X}\hat{\boldsymbol{\alpha}}(\delta)\right)-\mathbf{b}\right)^{\mathbf{T}}\boldsymbol{\omega}\right]$$

$$=E\left[-\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right.$$

$$\left.\left(\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{X}\left(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}(\delta)\right)+\mathbf{Zb}+\mathbf{e}\right)-\mathbf{b}\right)^{\mathbf{T}}\boldsymbol{\omega}\right]$$

$$=E\left[-\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right.$$

$$\left.\left(\mathbf{G}\mathbf{Z}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{X}.-\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})+\mathbf{Zb}+\mathbf{e}\right)-\mathbf{b}\right)^{\mathbf{T}}\boldsymbol{\omega}\right]$$

$$=E\left[-\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right.$$

$$\left.\left(\left(-\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})+(\mathbf{Zb}+\mathbf{e})\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}-\mathbf{b}^{\mathbf{T}}\right)\boldsymbol{\omega}\right]$$

$$=E\left[-\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right.$$

$$\left.\left(\left(-(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}+(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\right)\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}-\mathbf{b}^{\mathbf{T}}\right)\boldsymbol{\omega}\right]$$

$$=E\left[\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right.$$

$$(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}-$$

$$\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}+$$

$$\left.\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\mathbf{b}^{\mathbf{T}}\boldsymbol{\omega}\right]$$

$$=E\left[\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\right.$$

$$\left.\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}\right]-$$

$$E\left[\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}\right]+$$

$$E\left[\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\mathbf{b}^{\mathbf{T}}\boldsymbol{\omega}\right]$$

$$=\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\right]$$

$$\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}-$$

$$\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^{\mathbf{T}}\right]\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}+$$

$$\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}E\left[(\mathbf{Zb}+\mathbf{e})\mathbf{b}^{\mathbf{T}}\right]\boldsymbol{\omega}$$

$$= \mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}$$
$$\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega} -$$
$$\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega} +$$
$$\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}$$
$$= \mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega} -$$
$$\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega} +$$
$$\mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}$$
$$= \mathbf{X}\left(\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}^{\mathbf{T}}\boldsymbol{\omega}$$

III. $$E\left[(\mathbf{Zb}+\mathbf{e})\left(\hat{\boldsymbol{\alpha}}(\delta)-\boldsymbol{\alpha}\right)^{T}\boldsymbol{\lambda}\right] = E\left[(\mathbf{Zb}+\mathbf{e})\left(\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right)^{T}\boldsymbol{\lambda}\right]$$
$$= E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}\right]$$
$$= E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^{T}\right]\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$
$$= \boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$
$$= \mathbf{X}\left(\mathbf{X}^{T}\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

IV. $$\mathrm{E}\left[(\mathbf{Zb}+\mathbf{e})\left(\mathbf{GZ}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\mathbf{X}\hat{\alpha}(\delta)\right)-\mathbf{b}\right)^T\boldsymbol{\omega}\right]=E\left[(\mathbf{Zb}+\mathbf{e})\left(\mathbf{GZ}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{X}\boldsymbol{\alpha}+\mathbf{Zb}+\mathbf{e}-\mathbf{X}\hat{\alpha}(\delta)\right)-\mathbf{b}\right)^T\boldsymbol{\omega}\right]$$

$$=E\left[(\mathbf{Zb}+\mathbf{e})\left(\mathbf{GZ}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{X}\left(\boldsymbol{\alpha}-\hat{\alpha}(\delta)\right)+(\mathbf{Zb}+\mathbf{e})\right)-\mathbf{b}\right)^T\boldsymbol{\omega}\right]$$

$$=E\left[(\mathbf{Zb}+\mathbf{e})\left(\mathbf{GZ}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\boldsymbol{\alpha}-\hat{\alpha}(\delta)\right)+\mathbf{GZ}^T\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})-\mathbf{b}\right)^T\boldsymbol{\omega}\right]$$

$$=E\left[(\mathbf{Zb}+\mathbf{e})\left(\left(\boldsymbol{\alpha}-\hat{\alpha}(\delta)\right)^T\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T+(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T-\mathbf{b}^T\right)\boldsymbol{\omega}\right]$$

$$=E\Big[(\mathbf{Zb}+\mathbf{e})\left(\boldsymbol{\alpha}-\hat{\alpha}(\delta)\right)^T\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}+(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}-$$
$$(\mathbf{Zb}+\mathbf{e})\mathbf{b}^T\boldsymbol{\omega}\Big]$$

$$=E\Big[(\mathbf{Zb}+\mathbf{e})\left(-\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}(\mathbf{Zb}+\mathbf{e})\right)^T\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}+$$
$$(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}-(\mathbf{Zb}+\mathbf{e})\mathbf{b}^T\boldsymbol{\omega}\Big]$$

$$=E\Big[-(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}+$$
$$(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}-(\mathbf{Zb}+\mathbf{e})\mathbf{b}^T\boldsymbol{\omega}\Big]$$

$$=-E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}\right]+$$
$$E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}\right]-E\left[(\mathbf{Zb}+\mathbf{e})\mathbf{b}^T\boldsymbol{\omega}\right]$$

$$=-E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\right]\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}+$$
$$E\left[(\mathbf{Zb}+\mathbf{e})(\mathbf{Zb}+\mathbf{e})^T\right]\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}-E\left[(\mathbf{Zb}+\mathbf{e})\mathbf{b}^T\right]\boldsymbol{\omega}$$

$$=-\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}+$$
$$\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}-\mathbf{ZG}^T\boldsymbol{\omega}$$

$$=-\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}$$

Dari I, II, III, dan IV, maka:

$$\operatorname{cov}\left[\left(\hat{\tau}(\hat{\delta})-\hat{\tau}(\delta)\right),\left(\hat{\tau}(\delta)-t\right)\right]=-\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}+\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}+$$
$$\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\boldsymbol{\lambda}-\mathbf{X}\left(\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega}$$
$$=0$$

Karena $\text{cov}\left[\left(\hat{\tau}\left(\hat{\delta}\right)-\hat{\tau}(\delta)\right),\left(\hat{\tau}(\delta)-t\right)\right]=0$ maka $\hat{\tau}(\delta)-t$ berdistribusi

independent dengan $\hat{\tau}\left(\hat{\delta}\right)-\hat{\tau}(\delta)$, sehingga:

$$\mathbf{MSE}\left[\hat{\tau}\left(\hat{\delta}\right)\right]=\mathbf{MSE}\left[\hat{\tau}(\delta)\right]+E\left[\hat{\tau}\left(\hat{\delta}\right)-\hat{\tau}(\delta)\right]^2$$

LAMPIRAN 14

Menunjukkan bahwa dengan menggunakan Aproksimasi Taylor maka

$$E\left[\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}(\boldsymbol{\delta})\right]^2 \approx tr\left[\left(\partial\mathbf{a}^T/\partial\boldsymbol{\delta}\right)\boldsymbol{\Omega}\left(\partial\mathbf{a}^T/\partial\boldsymbol{\delta}\right)^T I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$
$$=: g_3(\boldsymbol{\delta})$$

Akan dtunjukkan $E\left[\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\tilde{\tau}(\boldsymbol{\delta})\right]^2 \approx tr\left[\left(\partial\mathbf{a}^T/\partial\boldsymbol{\delta}\right)\boldsymbol{\Omega}\left(\partial\mathbf{a}^T/\partial\boldsymbol{\delta}\right)^T I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right] =: g_3(\boldsymbol{\delta})$.

Dengan menggunakan pendekatan Taylor orde pertama dari $\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)$ di sekitar $\boldsymbol{\delta}$, maka :

$$\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right) \approx \hat{\tau}(\boldsymbol{\delta}) + \frac{\partial\hat{\tau}(\boldsymbol{\delta})}{\partial\boldsymbol{\delta}}\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right) \qquad \textbf{(L15.1)}$$

selanjutnya akan dicari bentuk $\left[\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}(\boldsymbol{\delta})\right]^2$. Berdasarkan persamaan **(L15.1)** maka $\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}(\boldsymbol{\delta}) \approx \frac{\partial\hat{\tau}(\boldsymbol{\delta})}{\partial\boldsymbol{\delta}}\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)$, kemudian dengan mengkuadratkan ruas bagian kiri dan kanan, maka diperoleh:

$$\left[\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}(\boldsymbol{\delta})\right]^2 \approx \left[\frac{\partial\hat{\tau}(\boldsymbol{\delta})}{\partial\boldsymbol{\delta}}\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\right]^2$$
$$\approx \left[\left(\frac{\partial\hat{\tau}(\boldsymbol{\delta})}{\partial\boldsymbol{\delta}}\right)^T\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\right]^2$$

Berdasarkan lampiran 10 diketahui bahwa

$$\hat{\tau}(\boldsymbol{\delta}, y) = \hat{\tau}^*(\boldsymbol{\delta}, \boldsymbol{\alpha}, y) + \mathbf{d}^T\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta})-\boldsymbol{\alpha}\right)$$

Maka

$$\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} = \frac{\partial\left(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta}, \boldsymbol{\alpha}, y) + \mathbf{d}^{T}\left(\hat{\boldsymbol{\alpha}}(\boldsymbol{\delta}) - \boldsymbol{\alpha}\right)\right)}{\partial \boldsymbol{\delta}},$$

dengan asumsi bahwa $\frac{\partial \mathbf{d}^{\mathrm{T}}}{\partial \boldsymbol{\delta}} \approx 0,$ didapatkan:

$$\begin{aligned}
\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} &\approx \frac{\partial\left(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta}, \boldsymbol{\alpha}, y)\right)}{\partial \boldsymbol{\delta}} \\
&\approx \frac{\partial\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\alpha} + \mathbf{a}^{\mathrm{T}}(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})\right)}{\partial \boldsymbol{\delta}} \\
&\approx \frac{\partial\left(\mathbf{a}^{\mathrm{T}}(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})\right)}{\partial \boldsymbol{\delta}} \\
&\approx \frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})
\end{aligned}$$

Sehingga,

$$\begin{aligned}
\left[\hat{\boldsymbol{\tau}}\left(\hat{\boldsymbol{\delta}}\right) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})\right]^{2} &\approx \left[\left(\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right)^{T}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)\right]^{2} \\
&\approx \left[\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y} - \mathbf{X}\boldsymbol{\alpha})\right)^{T}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)\right]^{2}
\end{aligned}$$

Kemudian dengan mengambil ekspektasinya, maka:

$$E\left[\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}\left(\boldsymbol{\delta}\right)\right]^2 \approx E\left[\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)^T \left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\right]^2$$

$$\approx tr\left[E\left(\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)\cdot\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\right)^T\right)\cdot E\left(\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T\right)\right]$$

$$\approx tr\left[E\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\cdot(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}\right)^T\right)\cdot I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$\approx tr\left[\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}E\left((\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})\cdot(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T\right)\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}\right)^T\cdot I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$\approx tr\left[\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}\cdot\boldsymbol{\Omega}\cdot\left(\frac{\partial \mathbf{a}^{\mathrm{T}}}{\partial \boldsymbol{\delta}}\right)^T\cdot I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$=: \mathrm{g}_3(\boldsymbol{\delta})$$

di mana

$$I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta}) = \mathrm{var}\left(\hat{\boldsymbol{\delta}}\right) = \left[-E\left(\frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\delta};\mathbf{y})}{\partial\delta_j\partial\delta_k}\right)\right]^{-1}$$, berdasarkan lampiran 9

$$= \left[\frac{1}{2}tr\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(j)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(k)}\right)\right]^{-1}$$

$$\therefore E\left[\hat{\tau}\left(\hat{\boldsymbol{\delta}}\right)-\hat{\tau}\left(\boldsymbol{\delta}\right)\right]^2 \approx \mathrm{g}_3(\boldsymbol{\delta})$$

LAMPIRAN 15

Menunjukkan $Eg_1\left(\hat{\boldsymbol{\delta}}\right) \approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$, $Eg_2\left(\hat{\boldsymbol{\delta}}\right) \approx g_2(\boldsymbol{\delta})$, dan

$Eg_3\left(\hat{\boldsymbol{\delta}}\right) \approx g_3(\boldsymbol{\delta})$

1. Akan ditunjukkan bahwa $Eg_1\left(\hat{\boldsymbol{\delta}}\right) \approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$

Berdasarkan lampiran 12 diketahui bahwa:

$$\begin{aligned} g_1(\boldsymbol{\delta}) &= \boldsymbol{\omega}^T\left(\mathbf{G} - \mathbf{G}\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}\right)\boldsymbol{\omega} \\ &= \boldsymbol{\omega}^T\mathbf{G}\boldsymbol{\omega} - \boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^T\boldsymbol{\Omega}^{-1}\mathbf{Z}\mathbf{G}\boldsymbol{\omega} \qquad \textbf{(L16.1)} \\ &= \boldsymbol{\omega}^T\mathbf{G}\boldsymbol{\omega} - \mathbf{a}^T\mathbf{Z}\mathbf{G}\boldsymbol{\omega} \end{aligned}$$

Kemudian untuk menunjukkan $Eg_1\left(\hat{\boldsymbol{\delta}}\right) \approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$, pertama

dibuat ekpansi Taylor untuk $g_1(\boldsymbol{\delta})$ di sekitar $\boldsymbol{\delta}$, yaitu:

$$g_1\left(\hat{\boldsymbol{\delta}}\right) = g_1(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_1(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_1(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right) + r\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right) \qquad \textbf{(L16.2)}$$

Dimana:

$\nabla g_1(\boldsymbol{\delta})$ adalah vektor dari turunan pertama $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$,

$\nabla^2 g_1(\boldsymbol{\delta})$ adalah matriks dari turunan kedua $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$, dan

$\lim\limits_{\hat{\boldsymbol{\delta}} \to \boldsymbol{\delta}} \frac{r\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)}{\left\|\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right\|^2} = 0$, sehingga **(L16.2)** menjadi:

$$g_1\left(\hat{\boldsymbol{\delta}}\right) \approx g_1(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_1(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_1(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right) \qquad \textbf{(L16.3)}$$

Berdasarkan bentuk $g_1(\boldsymbol{\delta})$ pada persamaan **(L16.1)** di atas, maka turunan

pertama $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$ adalah:

$$\frac{\partial g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_d} = \boldsymbol{\omega}^T \frac{\partial \mathbf{G}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_d}\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} - \mathbf{a}^T \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)$$

dengan

$$\begin{aligned}\mathbf{a} &= \boldsymbol{\Omega}^{-1}\mathbf{ZG}^T\boldsymbol{\omega} \\ &= \boldsymbol{\Omega}^{-1}\mathbf{ZG}\boldsymbol{\omega}\end{aligned}$$

Setelah turunan pertama $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$ didapat, selanjutnya akan dicari turunan kedua $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$, yaitu:

$$\begin{aligned}\frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} &= \boldsymbol{\omega}^T \frac{\partial \mathbf{G}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\right)^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \\ &\quad \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \mathbf{a}^T \frac{\partial}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) \\ &= \boldsymbol{\omega}^T \frac{\partial \mathbf{G}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\right)^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \\ &\quad \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \mathbf{a}^T \frac{\partial}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) \\ &= \underbrace{-\left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\right)^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}}_{i} \underbrace{- \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)}_{ii} \underbrace{- \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)}_{iii}\end{aligned}$$

dengan

$$\begin{aligned}\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d} &= \frac{\partial \boldsymbol{\Omega}^{-1}}{\partial \boldsymbol{\delta}_d}\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) \\ &= -\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right), \text{ dan} \\ &= -\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\end{aligned}$$

$$\frac{\partial^2 \mathbf{a}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} = \left(-\boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \frac{\partial \boldsymbol{\Omega}^{-1}}{\partial \boldsymbol{\delta}_e} - \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} - \frac{\partial \boldsymbol{\Omega}^{-1}}{\partial \boldsymbol{\delta}_e} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \right) \mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} -$$

$$\boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right) + \frac{\partial \boldsymbol{\Omega}^{-1}}{\partial \boldsymbol{\delta}_e} \frac{\partial}{\partial \boldsymbol{\delta}_d} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right) + \boldsymbol{\Omega}^{-1} \frac{\partial^2}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right)$$

$$= \left(\boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \right) \mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} -$$

$$\boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right) - \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right)$$

$$= \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} \mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} + \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} -$$

$$\boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right) - \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right)$$

$$= \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \mathbf{a} + \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \mathbf{a} -$$

$$\boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d} \boldsymbol{\Omega}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right) - \boldsymbol{\Omega}^{-1} \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e} \boldsymbol{\Omega}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} \left(\mathbf{ZG}(\boldsymbol{\delta}) \boldsymbol{\omega} \right)$$

Sehingga (i), (ii), dan (iii) menjadi:

$$
\begin{aligned}
(\mathbf{i}).\ -\left(\frac{\partial \mathbf{a}}{\partial\boldsymbol{\delta}_e\partial\boldsymbol{\delta}_d}\right)^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} = & -\Bigg[\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}+\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}- \\
& \boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial\boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)-\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial\boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\Bigg]^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} \\
= & -\Bigg[\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}+\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}- \\
& \frac{\partial}{\partial\boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}-\frac{\partial}{\partial\boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\Bigg]\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} \\
= & -\Bigg[\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}+\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}- \\
& \frac{\partial}{\partial\boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}-\frac{\partial}{\partial\boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}\Bigg] \\
= & \Bigg[-\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}-\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}+ \\
& \frac{\partial}{\partial\boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}+\frac{\partial}{\partial\boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}\Bigg] \\
= & \Bigg[-\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}-\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}+ \\
& \frac{\partial}{\partial\boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}+\frac{\partial}{\partial\boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}\Bigg] \\
= & \Bigg[-\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}-\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}+ \\
& \frac{\partial}{\partial\boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}+\frac{\partial}{\partial\boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\mathbf{a}\Bigg] \\
= & \Bigg[-2\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\mathbf{a}+\mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial\boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)+ \\
& \mathbf{a}^T\frac{\partial\boldsymbol{\Omega}}{\partial\boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial\boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\Bigg]
\end{aligned}
$$

$$(\mathbf{ii}).-\left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) = -\left[-\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right]^T \frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)$$

$$= -\left[-\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1} + \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\right]\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)$$

$$= \left[\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right]$$

$$(\mathbf{iii}).-\left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) = -\left[-\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right]^T \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)$$

$$= -\left[-\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1} + \frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\right]\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)$$

$$= \left[\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right]$$

$$= \left[\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right]$$

berdasarkan i, ii, dan iii maka $\frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}$ menjadi:

$$\frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} = \left[-2\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\mathbf{a} + \mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) + \right.$$

$$\left. \mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right] +$$

$$\left[\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right] +$$

$$\left[\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right]$$

$$= -2\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\mathbf{a} + 2\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) +$$

$$2\mathbf{a}^T \frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - 2\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)$$

Kemudian akan dicari nilai ekspektasi terhadap $g_1\left(\hat{\boldsymbol{\delta}}\right)$ **(L16.3),** yaitu sebagai berikut:

$$E\left[g_1\left(\hat{\boldsymbol{\delta}}\right)\right] \approx E\left[g_1(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T \nabla g_1(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T \nabla^2 g_1(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\right]$$

$$\approx E\left[g_1(\boldsymbol{\delta})\right] + E\left[\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T \nabla g_1(\boldsymbol{\delta})\right] + E\left[\frac{1}{2}\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T \nabla^2 g_1(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\right]$$

$$\approx E\left[g_1(\boldsymbol{\delta})\right] + E\left[\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T \nabla g_1(\boldsymbol{\delta})\right] + E\left[\frac{1}{2}\nabla^2 g_1(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T\right]$$

$$\approx g_1(\boldsymbol{\delta}) + E\left[\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T\right]\nabla g_1(\boldsymbol{\delta}) + E\left[\frac{1}{2}\nabla^2 g_1(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)^T\right]$$

$$\approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) + \frac{1}{2}tr\left[\nabla^2 g_1(\boldsymbol{\delta}) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})$ merupakan variansi dari $\boldsymbol{\delta}_{\mathbf{ML}}$.

Berdasarkan lampiran 14 , diperoleh bahwa adalah

$$g_3(\boldsymbol{\delta}) =: tr\left[\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}}\right)\boldsymbol{\Omega}\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}}\right)^T I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

Selanjutnya, jika bentuk dari $tr\left[\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}_{\mathrm{d}}}\right)\boldsymbol{\Omega}\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}_{\mathrm{e}}}\right)^T I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$ dijabarkan, maka akan diperoleh:

$$tr\left[\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}_\mathrm{d}}\right)\boldsymbol{\Omega}\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}_\mathrm{e}}\right)^T I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right] = tr\left[\left(-\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right)^{\mathbf{T}}\boldsymbol{\Omega}\right.$$

$$\left.\left(-\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\left(-\mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1} + \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\right)\boldsymbol{\Omega}\right.$$

$$\left.\left(-\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\left(-\mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega} + \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\right)\right.$$

$$\left.\left(-\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\left(-\mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\mathbf{I} + \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\mathbf{I}\right)\right.$$

$$\left.\left(-\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\left(\mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} - \mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \right.\right.$$

$$\left.\left.\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\left(\mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\mathbf{a} - \mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_d}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) - \right.\right.$$

$$\left.\left.\mathbf{a}^T\frac{\partial \boldsymbol{\Omega}}{\partial \boldsymbol{\delta}_e}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right) + \frac{\partial}{\partial \boldsymbol{\delta}_d}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}\left(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}\right)\right) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$= -\frac{1}{2}tr\left[\frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

Oleh karena $tr\left[\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}_\mathrm{d}}\right)\boldsymbol{\Omega}\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}_\mathrm{e}}\right)^T I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right] = -\frac{1}{2}tr\left[\frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$, maka

$g_3(\boldsymbol{\delta}) = -\frac{1}{2} tr\left[\nabla^2 g_1(\boldsymbol{\delta}) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$, sehingga:

$E\left[g_1\left(\hat{\boldsymbol{\delta}}\right)\right] \approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta}) \nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$.

$\therefore E\left[g_1\left(\hat{\boldsymbol{\delta}}\right)\right] \approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta}) \nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$

2. Akan ditunjukkan bahwa $Eg_2\left(\hat{\boldsymbol{\delta}}\right) \approx g_2(\boldsymbol{\delta})$

untuk menunjukkan $Eg_2\left(\hat{\boldsymbol{\delta}}\right) \approx g_2(\boldsymbol{\delta})$, pertama dibuat ekpansi Taylor untuk $g_2(\boldsymbol{\delta})$ di sekitar $\boldsymbol{\delta}$, yaitu:

$$g_2\left(\hat{\boldsymbol{\delta}}\right) = g_2(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_2(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_2(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right) + r\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)$$

Dimana:

$\nabla g_2(\boldsymbol{\delta})$ adalah vektor dari turunan pertama $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$,

$\nabla^2 g_2(\boldsymbol{\delta})$ adalah matriks dari turunan kedua $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$, dan

$\lim_{\hat{\boldsymbol{\delta}} \to \boldsymbol{\delta}} \frac{r\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)}{\left\|\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right\|^2} = 0$, sehingga

$$g_2\left(\hat{\boldsymbol{\delta}}\right) \approx g_2(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_2(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_2(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)$$

Diketahui bahwa:

$$g_2(\boldsymbol{\delta}) = \mathbf{d}^{\mathbf{T}}\left(\mathbf{X}^T \boldsymbol{\Omega}^{-1} \mathbf{X}\right)\mathbf{d}$$

dengan

$$\mathbf{d} = \boldsymbol{\lambda}^T - \boldsymbol{\omega}^T \mathbf{G} \mathbf{Z}^T \boldsymbol{\Omega}^{-1} \mathbf{X}$$

dan dengan asumsi bahwa

$\frac{\partial \mathbf{d}^{\mathrm{T}}}{\partial \boldsymbol{\delta}} \approx 0$ maka mengakibatkan $\nabla g_2(\boldsymbol{\delta}) \approx 0$, sehingga didapatkan:

$$E\left[g_2\left(\hat{\boldsymbol{\delta}}\right)\right] \approx E\left[g_2(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_2(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_2(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)\right]$$

$$\approx g_2(\boldsymbol{\delta})$$

$$\therefore E\left[g_2\left(\hat{\boldsymbol{\delta}}\right)\right] \approx g_2(\boldsymbol{\delta})$$

3. Akan ditunjukkan bahwa $Eg_3\left(\hat{\boldsymbol{\delta}}\right) \approx g_3(\boldsymbol{\delta})$.

untuk menunjukkan $Eg_3\left(\hat{\boldsymbol{\delta}}\right) \approx g_3(\boldsymbol{\delta})$, pertama dibuat ekpansi Taylor untuk $g_3(\boldsymbol{\delta})$

di sekitar $\boldsymbol{\delta}$, yaitu:

$$g_3\left(\hat{\boldsymbol{\delta}}\right) = g_3(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_3(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_3(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right) + r\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)$$

di mana:

$\nabla g_3(\boldsymbol{\delta})$ adalah vektor dari turunan pertama $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$,

$\nabla^2 g_3(\boldsymbol{\delta})$ adalah matriks dari turunan kedua $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$, dan

$\lim_{\hat{\boldsymbol{\delta}} \to \boldsymbol{\delta}} \frac{r\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)}{\left\|\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right\|^2} = 0$, sehingga

$$g_3\left(\hat{\boldsymbol{\delta}}\right) \approx g_3(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_3(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_3(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)$$

Selanjutnya diketahui bahwa:

$g_3(\boldsymbol{\delta}) = -\frac{1}{2} tr\left[\nabla^2 g_1(\boldsymbol{\delta}) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$, sehingga:

$$g_3\left(\hat{\boldsymbol{\delta}}\right) \approx g_3(\boldsymbol{\delta}) + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla g_3(\boldsymbol{\delta}) + \frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \nabla^2 g_3(\boldsymbol{\delta})\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)$$

$$\approx -\frac{1}{2} tr\left[\nabla^2 g_1(\boldsymbol{\delta}) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right] + \left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \left(-\frac{1}{2} tr\left[\nabla^3 g_1(\boldsymbol{\delta}) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]\right) +$$

$$\frac{1}{2}\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)^T \left(-\frac{1}{2} tr\left[\nabla^4 g_1(\boldsymbol{\delta}) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]\right)\left(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)$$

Berdasarkan bukti (1), maka:

$$g_3\left(\hat{\boldsymbol{\delta}}\right) \approx -\frac{1}{2} tr\left[\nabla^2 g_1(\boldsymbol{\delta}) I_{\boldsymbol{\delta}}^{-1}(\boldsymbol{\delta})\right]$$

$$\approx g_3(\boldsymbol{\delta})$$

$$\therefore E\left[g_3\left(\hat{\boldsymbol{\delta}}\right)\right] \approx g_3(\boldsymbol{\delta})$$

LAMPIRAN 16

Pengujian Kenormalan

Tests of Normality

	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
pendapatan	.144	15	.200*	.940	15	.386

*. This is a lower bound of the true significance.

a. Lilliefors Significance Correction

LAMPIRAN 17

Penaksiran Parameter pada *General Linear Mixed Model* Menggunakan *Software* MATLAB 7.0.1.

```
clear;
tic;
syms d0 d1;
X = [0 0 1; 0 1 0; 0 1 0; 0 1 0; 1 0 0;...
     1 0 0; 1 0 0; 1 0 0; 0 0 1; 0 0 1;...
     0 0 1; 1 0 0; 0 1 0; 0 1 0; 0 0 1];
Y = [267.80; 248.84; 247.89; 251.21; 276.48;...
     229.77; 231.92; 232.52; 219.36; 212.42;...
     181.26; 201.34; 143.37; 144.46; 186.87];
Z = [1 0 0; 1 0 0; 1 0 0; 1 0 0; 1 0 0;...
     0 1 0; 0 1 0; 0 1 0; 0 1 0; 0 1 0;...
     0 0 1; 0 0 1; 0 0 1; 0 0 1; 0 0 1];
delta = [0.01; 0.01];
epsilon = [0.01; 0.01];
err = [1; 1];

R = d0*eye(15);
G = d1*eye(3);
O = R + Z*G*Z';
a_cap = inv(X'*inv(O)*X)*(X'*inv(O)*Y);
diffy(1,:,:) = diff(O, d0);
diffy(2,:,:) = diff(O, d1);
```

```
for j=1:2
    p = diffy(j,:,:);
    p = reshape(p, 15, 15);
    S(j) = -1/2*trace(inv(O)*p) + 1/2*(Y - X*a_cap)'*...
          (inv(O)*p*inv(O))*(Y-X*a_cap);
    for k=1:2
        q = diffy(k,:,:);
        q = reshape(q, 15, 15);
        L(j,k) = 1/2*trace((inv(O)*p) * (inv(O)*q));
    end
end

while norm(err,inf) >= epsilon
    nS = subs(S, {d0, d1}, {delta(1), delta(2)});
    nL = subs(L, {d0, d1}, {delta(1), delta(2)});

    del = delta;
    delta = del + inv(nL)*nS';

    err = abs(delta - del);
 end

 R = delta(1)*eye(15);
 G = delta(2)*eye(3);
 O = R + Z*G*Z';
 a_cap = inv(X'*inv(O)*X)*(X'*inv(O)*Y)
 b_cap = G*Z'*inv(O)*(Y - X*a_cap)
 y_cap = X*a_cap + Z*b_cap
```

```
Y - y_cap

t = toc;

fprintf('Waktu: %f', t);
```

LAMPIRAN 18

Tabel Perbandingan antara $\mathbf{y}$ dan $\hat{\mathbf{y}}$.

Pendapatan dari Data	**Pendapatan dari Taksiran**
267.8	272.6338
248.84	244.0388
247.89	244.0388
251.21	244.0388
276.48	286.1588
229.77	230.6414
231.92	230.6414
232.52	230.6414
219.36	217.1164
212.42	217.1164
181.26	180.4218
201.34	193.9468
143.37	151.8268
144.46	151.8268
186.87	180.4218

LAMPIRAN 19

Tabel *Error*.

Error
-4.8338
4.8012
3.8512
7.1712
-9.6788
-0.8714
1.2786
1.8786
2.2436
-4.6964
0.8382
7.3932
-8.4568
-7.3668
6.4482

LAMPIRAN 20

Penaksiran *Mean Squared Error* (MSE) *Empirical Best Linear Unbiased Prediction* (EBLUP) pada *General Linear Mixed Model* Menggunakan *Software* MATLAB 7.0.1.

```
clear;
syms d0 d1;
L = [0 ; 1; 0];
Q = [0; 0; 1; 0; 0; 0];
X = [0 0 1; 0 1 0; 0 1 0; 0 1 0; 1 0 0;...
     1 0 0; 1 0 0; 1 0 0; 0 0 1; 0 0 1;...
     0 0 1; 1 0 0; 0 1 0; 0 1 0; 0 0 1];
Z = [1 0 0; 1 0 0; 1 0 0; 1 0 0; 1 0 0;...
     0 1 0; 0 1 0; 0 1 0; 0 1 0; 0 1 0;...
     0 0 1; 0 0 1; 0 0 1; 0 0 1; 0 0 1];
Y = [267.80; 248.84; 247.89; 251.21; 276.48;...
     229.77; 231.92; 232.52; 219.36; 212.42;...
     181.26; 201.34; 143.37; 144.46; 186.87];
R = d0*eye(15);
G = d1*eye(3);
O = R + Z*G*Z';
e = L' - Q'*G*Z'*inv(O)*X;

g2 = e*inv(X'*inv(O)*X)*e';

g1 = Q'*(G - G*Z'*inv(O)*Z*G)*Q;
G1(1,:,:) = diff(g1, d0);
```

```
G1(2,:,:) = diff(g1, d1);


u = Q'*G*Z'*inv(O);
diffO(1,:,:) = diff(O, d0);
diffO(2,:,:) = diff(O, d1);
diffG(1,:,:) = diff(G, d0);
diffG(2,:,:) = diff(G, d1);


for j=1:2
    p = diffO(j,:,:);
    p = reshape(p, 15, 15);
    q = diffG(j,:,:);
    q = reshape(q, 3, 3);
    U(j,:) = -inv(O)*p*u' + inv(O)*Z*q*Q;
end


ib = X'*inv(O)*X;
IB(1,:,:) = diff(ib, d0);
IB(2,:,:) = diff(ib, d1);


a_cap = inv(X'*inv(O)*X)*(X'*inv(O)*Y);
diffid(1,:,:) = diff(O, d0);
diffid(2,:,:) = diff(O, d1);
for j=1:2
    s = diffid(j,:,:);
    s = reshape(s, 15, 15);
    S(j) = -1/2*trace(inv(O)*s) + 1/2*(Y - X*a_cap)'*...
           (inv(O)*s*inv(O))*(Y-X*a_cap);
```

```
    for k=1:2
        t = diffid(k,:,:);
        t = reshape(t, 15, 15);
        ID(j,k) = 1/2*trace((inv(O)*s) * (inv(O)*t));
    end
end

g3 = trace(U*O*U'*inv(ID));

b = 1/2*inv(ID)*[trace(inv(ib)*IB(1)); trace(inv(ib)*IB(2))];

MSE = g1 - b'*G1 + g2 + 2*g3;
nMSE = subs(MSE, {d0, d1}, {38.5, 1444.5})
```