CHAPTER III

THE PROOF OF INEQUALITIES

In this Chapter, the main purpose is to prove four theorems about Hardy-Littlewood-Pólya Inequality and then gives some examples of their application. We will begin with Classical Hardy-Littlewood-Pólya Majorization Inequality and then prove its Riemann form; furthermore the extended inequality from **[1]** will be stated together with its Riemann form.

3.1 The Classical Hardy-Littlewood- Pólya Majorization Inequality and its Riemann Integral Form

Before we state the inequality, we need to introduce some knowledge about majorization theory. This concept has been studied several times in the past through study of matrices, vector and stochastic process by some mathematicians such as Karamata, Muirhead, G. Hardy, J.E. Littlewood & G.Pólya. We first give the definition of majorization for decreasing arrangement array and extend the definition for arbitrary array.

Definition 3.3.1

Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ be any array of real numbers such that $x_i \geq x_{i+1}$ and $y_i \geq y_{i+1}$ for $i = 1..n-1$. We say that $\boldsymbol{x}$ *majorizes* $\boldsymbol{y}$ (abbreviated $\boldsymbol{x} \succ \boldsymbol{y}$) if and only if

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \qquad k = 1,2,\cdots,n-1$$

And

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Let $\boldsymbol{u} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{v} = (y_1, y_2, \cdots, y_n)$ be any array of real numbers. Let $\{x_i^*\}_{i=1}^{n}$ and $\{y_i^*\}_{i=1}^{n}$ be rearrangement of $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ respectively in decreasing order, then $\boldsymbol{u} \succ \boldsymbol{v}$ if $(x_1^*, x_2^*, \cdots, x_n^*) \succ (y_1^*, y_2^*, \cdots, y_n^*)$.

There are many summation involved when working with majorization inequality, therefore we need a formula concerning to the sum. One of them is a well-known formula due to Abel, the Abel summation formula, or summation by part. This is an analog form of integration by part for summation.

Theorem 3.1.1 (Abel Summation Formula)

If $\{a_i\}_{i=1}^{n}$ *and* $\{b_i\}_{i=1}^{n}$ *are two sequences, define*

$$A_m = \sum_{k=1}^{m} a_k \quad \textit{for} \quad m \geq 1$$

and $A_0 = 0$ *then for any* $1 \leq p \leq q$ *we have*

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p$$

Proof:

Since $a_m = A_m - A_{m-1}$ for any $m \geq 1$, we have

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} (A_k - A_{k-1}) b_k = \sum_{k=p}^{q} A_k b_k - \sum_{k=p}^{q} A_{k-1} b_k$$

$$= \sum_{k=p}^{q} A_k b_k - \sum_{k=p-1}^{q-1} A_k b_{k+1}$$

$$= \sum_{k=p}^{q-1} A_k b_k + A_q b_q - \left(\sum_{k=p}^{q-1} A_k b_{k+1} + A_{p-1} b_p \right)$$

and thus

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p$$

as desired.■

The proof of Hardy-Littlewood- Pólya Majorization Inequality will be established by first proving the following lemma.

Lemma 3.1.1 Let $\{x_i\}_{i=1}^{n}$ *and* $\{y_i\}_{i=1}^{n}$ *be two nonincreasing sequences of real numbers. Define*

$$z_i = \begin{cases} \dfrac{f(x_i) - f(y_i)}{x_i - y_i}, & x_i \neq y_i \\ f'_{+}(x_i), & x_i = y_i. \end{cases}$$

If f *is convex function then the sequence* $\{z_i\}_{i=1}^{n}$ *is a nonincreasing sequence.*

Proof:

Let f be a convex function, for any two nonincreasing sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ the following cases may occur:

I. $y_i = x_i$ and $y_{i+1} = x_{i+1}$.

II. $y_i \neq x_i$, $x_{i+1} \neq y_i$ and $y_{i+1} \neq x_{i+1}$.

III. $y_i = x_i$, $x_{i+1} \neq y_i$ and $y_{i+1} \neq x_{i+1}$.

IV. $y_i = x_i$, $x_{i+1} = y_i$ and $y_{i+1} \neq x_{i+1}$.

V. $y_i \neq x_i$, $x_{i+1} \neq y$ and $y_{i+1} = x_{i+1}$.

VI. $y_i \neq x_i$, $x_{i+1} = y_i$ and $y_{i+1} = x_{i+1}$.

<u>Case I:</u> Since $y_i = x_i$ and $y_{i+1} = x_{i+1}$ then $z_i = f'_+(x_i) \geq f'_+(x_{i+1}) = z_{i+1}$, by Theorem 2.1.6.

<u>Case II:</u> Since $x_{i+1} \leq x_i$ then for $y_i \notin \{x_i, x_{i+1}\}$, we can apply Galvani's lemma

$$z_i = \frac{f(x_i) - f(y_i)}{x_i - y_i} \geq \frac{f(x_{i+1}) - f(y_i)}{x_{i+1} - y_i} = \frac{f(y_i) - f(x_{i+1})}{y_i - x_{i+1}}$$

and since $y_{i+1} \leq y_i$ then for $y_{i+1} \neq x_{i+1}$, we have

$$\frac{f(y_i) - f(x_{i+1})}{y_i - x_{i+1}} \geq \frac{f(y_{i+1}) - f(x_{i+1})}{y_{i+1} - x_{i+1}} = z_{i+1}$$

so $z_i \geq z_{i+1}$.

<u>Case III:</u> Let $\epsilon > 0$, then $x_{i+1} < x_i + \epsilon$ and $y_i < x_i + \epsilon$, and we have

$$\frac{f(x_i + \epsilon) - f(y_i)}{(x_i + \epsilon) - y_i} \geq \frac{f(x_{i+1}) - f(y_i)}{x_{i+1} - y_i} = \frac{f(y_i) - f(x_{i+1})}{y_i - x_{i+1}}$$

Since $y_{i+1} \leq x_{i+1}$ then

$$\frac{f(y_i) - f(x_{i+1})}{y_i - x_{i+1}} \geq \frac{f(y_{i+1}) - f(x_{i+1})}{y_{i+1} - x_{i+1}} = z_{i+1}$$

So

$$\frac{f(x_i + \epsilon) - f(y_i)}{(x_i + \epsilon) - y_i} \geq z_{i+1}$$

As $\epsilon \to 0^+$ we have

$$z_i = f_+'(x_i) = \lim_{\epsilon \to 0^+} \frac{f(x_i + \epsilon) - f(y_i)}{(x_i + \epsilon) - y_i} \geq z_{i+1}$$

<u>Case IV:</u> Suppose that $x_{i+1} < y_{i+1}$, then $y_i = x_{i+1} < y_{i+1}$, which is a contradiction. So we should have $x_{i+1} > y_{i+1}$.

Since $x_i \geq x_{i+1}$ we have $x_i > y_{i+1}$. Take $\epsilon > 0$ such that $x_i + \epsilon > y_{i+1}$ then

$$\frac{f(x_i + \epsilon) - f(y_i)}{(x_i + \epsilon) - y_i} \geq \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i} = \frac{f(x_{i+1}) - f(y_{i+1})}{x_{i+1} - y_{i+1}}$$

As $\epsilon \to 0^+$ we have $z_i \geq z_{i+1}$.

<u>Case V:</u> Since $x_{i+1} \leq x_i$ then for $y_i \neq x_{i+1}$

$$z_i = \frac{f(x_i) - f(y_i)}{x_i - y_I} \geq \frac{f(x_{i+1}) - f(y_i)}{x_{i+1} - y_i} = \frac{f(y_i) - f(x_{i+1})}{y_i - x_{i+1}}$$

since $y_i \neq x_{i+1}$ and $y_{i+1} \leq y_i$ then $y_{i+1} < y_i$ so for sufficiently small $\epsilon > 0$ we have $y_{i+1} + \epsilon < y_i$ and hence

$$\frac{f(y_i) - f(x_{i+1})}{y_i - x_{i+1}} \geq \frac{f(y_{i+1} + \epsilon) - f(y_{i+1})}{(y_{i+1} + \epsilon) - x_{i+1}}$$

so

$$z_i = \frac{f(x_i) - f(y_i)}{x_i - y_I} \geq \frac{f(y_{i+1} + \epsilon) - f(x_{i+1})}{(y_{i+1} + \epsilon) - x_{i+1}}$$

as $\epsilon \to 0^+$ we conclude that $z_i \geq f'_+(x_{i+1}) = z_{i+1}$.

Case vi) since $x_i \neq y_i$ and $x_i \geq x_{i+1}$ then $x_i > x_{i+1}$ so for sufficiently small $\epsilon > 0$ we have $x_i > x_{i+1} + \epsilon$ and hence since $y_i = x_{i+1} = y_{i+1}$ we have

$$z_i = \frac{f(x_i) - f(y_i)}{x_i - y_I} \geq \frac{f(x_{i+1} + \epsilon) - f(y_i)}{(x_{i+1} + \epsilon) - y_i} = \frac{f(y_i) - f(x_{i+1} + \epsilon)}{y_i - (x_{i+1} + \epsilon)} = \frac{f(y_{i+1}) - f(x_{i+1} + \epsilon)}{y_{i+1} - (x_{i+1} + \epsilon)}$$

As $\epsilon \to 0^+$ we have $z_i \geq f'_+(x_{i+1}) = z_{i+1}$ as well■.

The Hardy-Littlewood- Pólya Majorization Inequality has several forms. The classical form which will be given here, is the simplest from the others which also can be found in **[3]**, and this classical inequality also founded independently by Karamata, thereby some literatures state it as Karamata Inequality.

Theorem 3.1.2 (The Classical Hardy-Littlewood- Pólya Majorization Inequality)

Let $f: X \to \mathbb{R}$ be a convex function, and suppose that $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ such that $\boldsymbol{x} \succ \boldsymbol{y}$, and then we have

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i) \qquad (3.1.1)$$

Proof:

Without loss of generality, it is enough for us to prove the theorem when $\boldsymbol{x}$ and $\boldsymbol{y}$ are replaced by $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ respectively, since

$$\sum_{i=1}^{n} f(x_i^*) = \sum_{i=1}^{n} f(x_i) \quad \text{and} \quad \sum_{i=1}^{n} f(y_i^*) = \sum_{i}^{n} f(y_i)$$

Define the sequence $\{z_i\}_{i=1}^{n}$ by

$$z_i := \begin{cases} \dfrac{f(x_i^*) - f(y_i^*)}{x_i^* - y_i^*}, & x_i^* \neq y_i^* \\ f'_+(x_i^*), & x_i^* = y_i^* \end{cases}$$

Since $\{x_i^*\}_{i=1}^{n}$ and $\{y_i^*\}_{i=1}^{n}$ are nonincreasing, then z_i also nonincreasing by lemma 3.1.1. Notice that by Abel Summation Formula, for $A_m = \sum_{i=1}^{m} x_i^* - y_i^*$ and $A_0 = 0$ we have

$$\sum_{i=1}^{n} f(x_i^*) - \sum_{i=1}^{n} f(y_i^*) = \sum_{i=1}^{n} (x_i^* - y_i^*)\, z_i$$

$$= \sum_{i=1}^{n-1} A_i(z_i - z_{i+1}) + A_n z_n - A_0 z_1$$

$$= \sum_{i=1}^{n-1} A_i(z_i - z_{i+1}) + A_n z_n$$

Let $U_k = \sum_{j=1}^{k} x_j^*$ and $V_k = \sum_{j=1}^{k} y_j^*$ for $k = 1,2,\cdots,n$, we have $A_m = U_m - V_m$ then the above equation can be written in the form

$$\sum_{i=1}^{n} f(x_i^*) - \sum_{i=1}^{n} f(y_i^*) = \sum_{i=1}^{n-1} (z_i - z_{i+1})(U_i - V_i) + z_n(U_n - V_n) \qquad (3.1.2)$$

The condition $\boldsymbol{x} \succ \boldsymbol{y}$ can be viewed as $U_i \geq V_i$ and $U_n = V_n$. And since $z_i - z_{i+1} \geq 0$ we have

$$\sum_{i=1}^{n} f(x_i^*) - \sum_{i=1}^{n} f(y_i^*) \geq 0$$

which is equivalent to

$$\sum_{i=1}^{n} f(x_i^*) \geq \sum_{i=1}^{n} f(y_i^*) \qquad \blacksquare.$$

The inequality in Theorem 3.1.2 can be extended to Riemann integral analog form, which also will be proved here. Before we proceed to prove the integral form, we first prove the following lemmas:

Lemma 3.1.2 *Let $\varphi: [a,b] \to X$ be a continuous function, then for any $[c,d] \subset [a,b]$ with $c \neq d$, there exist $\xi \in [c,d]$ such that $\int_c^d \varphi(t)dt = (d-c)\varphi(\xi)$.*

Proof:

Since φ is continuous on $[a,b]$, it is also continuous on $[c,d] \subset [a,b]$ thus φ attains its maximum and minimum on $[c,d]$ say $\min_{t\in[c,d]} \varphi(t) = \varphi(x_1)$ and $\max_{t\in[c,d]} \varphi(t) = \varphi(x_2)$ where $x_1, x_2 \in [c,d]$ and we have $\varphi(x_1) \leq \varphi(t) \leq \varphi(x_2)$ for $t \in [c,d]$, integrating from c to d we have

$$(d-c)\varphi(x_1) \leq \int_c^d \varphi(t)dt \leq (d-c)\varphi(x_2).$$

Equivalently, because $c \neq d$ we have

$$\varphi(x_1) \leq \left(\frac{1}{d-c}\right)\int_c^d \varphi(t)dt \leq \varphi(x_2).$$

And by continuity of φ, the Intermediate Value Theorem guaranties that there exist $\xi \in [x_1, x_2] \subset [c, d]$ such that $\varphi(\xi) = \left(\frac{1}{d-c}\right)\int_c^d \varphi(t)dt$, and the lemma is proved. ■

Lemma 3.1.3. *Let* $\phi, \varphi: [a, b] \to X$ *are continuous and monotone decreasing functions on* $[a, b]$ *and satisfies:*

i) $\int_a^x \varphi(t)dt \geq \int_a^x \phi(t)dt$ *for any* $x \in [a, b]$

ii) $\int_a^b \varphi(t)dt = \int_a^b \phi(t)dt$

Let $x_k = a + \frac{(b-a)k}{n}$*, then there exist the sequences* $c_k \in [x_{k-1}, x_k]$ *and* $d_k \in [x_{k-1}, x_k]$ *such that*

$$\big(\varphi(c_1), \varphi(c_2), \cdots, \varphi(c_n)\big) \succ (\phi(d_1), \phi(d_2), \cdots, \phi(d_n))$$

Proof:

By the lemma 3.1.2, since φ is continuous there exist $c_k \in [x_{k-1}, x_k]$ such that

$$\int_{x_{k-1}}^{x_k} \varphi(x) = (x_k - x_{k-1})\varphi(c_k) = \left(\frac{b-a}{n}\right)\varphi(c_k).$$

Similarly there exist $d_k \in [x_{k-1}, x_k]$ such that

$$\int_{x_{k-1}}^{x_k} \phi(x) = (x_k - x_{k-1})\phi(d_k) = \left(\frac{b-a}{n}\right)\phi(d_k).$$

From condition (i) and since $x_0 = a$ we have for $m = 1,2, \cdots, n-1$

$$\sum_{k=1}^{m}\varphi(c_k)=\left(\frac{n}{b-a}\right)\int_a^{x_m}\varphi(x)dx\geq\left(\frac{n}{b-a}\right)\int_a^{x_m}\phi(x)dx=\sum_{k=1}^{m}\phi(d_k). \qquad (3.1.4)$$

Moreover, for $m=n$ from condition (ii) and since $x_n=b$ we have

$$\sum_{k=1}^{n}\varphi(c_k)=\left(\frac{n}{b-a}\right)\int_a^{b}\varphi(x)dx=\left(\frac{n}{b-a}\right)\int_a^{b}\phi(x)dx=\sum_{k=1}^{n}\phi(d_k). \qquad (3.1.5)$$

By definition of majorization since both ϕ and φ are decreasing, (3.1.4) and (3.1.5) is

$$\big(\varphi(c_1),\varphi(c_2),\cdots,\varphi(c_n)\big)\succ\big(\phi(d_1),\phi(d_2),\cdots,\phi(d_n)\big) \quad \blacksquare$$

We now attempt to prove the Riemann Integral analog form of Theorem 3.1.2. This analog version replaces the array in Theorem 3.1.2 by monotone continuous function and gives us an idea of majorization of two functions, which will be presented by integral inequality.

Theorem 3.1.3 (The Riemann form of Hardy-Littlewood-Pólya Majorization inequality)

Let $\phi,\varphi:[a,b]\to X$ *are continuous, onto and monotone decreasing functions on* $[a,b]$ *and satisfies:*

i) $\int_a^x\varphi(t)dt\geq\int_a^x\phi(t)dt$ *for any* $x\in[a,b]$

ii) $\int_a^b\varphi(t)dt=\int_a^b\phi(t)dt.$

If f is a convex function, then

$$\int_a^b f(\varphi(t))dt\geq\int_a^b f(\phi(t))dt$$

Proof:

Since i) and ii) are satisfied then by lemma 3.1.3, there exist sequences $\{c_k\}_{k=1}^{n}$ and $\{d_k\}_{k=1}^{n}$ such that

$$\big(\varphi(c_1), \varphi(c_2), \cdots, \varphi(c_n)\big) \succ \big(\phi(d_1), \phi(d_2), \cdots, \phi(d_n)\big).$$

From the convexity of f, by Theorem 3.1.2 we have

$$\sum_{k=1}^{n} f(\varphi(c_k)) \geq \sum_{k=1}^{n} f\big(\phi(d_k)\big) \tag{3.1.6}$$

Multiplying both sides of equation 3.1.6 by $\frac{b-a}{n}$, we have

$$\sum_{k=1}^{n} f(\varphi(c_k)) \left(\frac{b-a}{n}\right) \geq \sum_{k=1}^{n} f\big(\phi(d_k)\big) \left(\frac{b-a}{n}\right) \tag{3.1.7}$$

each $c_k, d_k \in [x_{k-1}, x_k]$ with $x_k = a + \frac{(b-a)k}{n}$. We know that both ϕ and φ are continuous functions that map closed interval $[a, b]$ onto X, thus X is also a closed interval, and hence by Theorem 2.1.3 since f is convex then f is continuous. Thus $f \circ \phi$ and $f \circ \varphi$ are also continuous, hence they are Riemann integrable, thus both limits $\lim_{n\to\infty} \sum_{k=1}^{n} f(\varphi(c_k)) \left(\frac{b-a}{n}\right)$ and $\lim_{n\to\infty} \sum_{k=1}^{n} f(\phi(c_k)) \left(\frac{b-a}{n}\right)$ exist and equal to their correspond integral and inequality 3.1.6 becomes

$$\int_a^b f(\varphi(t))dt \geq \int_a^b f(\phi(t))dt \qquad \blacksquare$$

3.2 The Extension of Hardy-Littlewood- Pólya Majorization Inequality and Its Integral Riemann Form.

The next inequality that will be proved in this section is the generalization of Theorem 3.1.2 which makes use the relative convexity as a bridge of generalization. This generalization is the result of C.P. Nisculescu and F. Popovici on their joint paper[3].

Theorem 3.2.1 (The Generalization Of Hardy Littlewood Majorization Inequality)

Let $f, g: X \to \mathbb{R}$ be two functions such that $g \lhd f$ and let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ are in X. If for any real weights $p_1, p_2, \cdots, p_n$ all three conditions below is satisfied:

i) $g(x_1) \geq \cdots \geq g(x_n)$ *and* $g(y_1) \geq \cdots \geq g(y_n)$

ii) $\sum_{k=1}^{r} p_k g(x_k) \geq \sum_{k=1}^{r} p_k g(y_k)$ *for all* $r = 1, \cdots, n$

iii) $\sum_{k=1}^{r} p_k g(x_k) = \sum_{k=1}^{r} p_k g(y_k)$

then

$$\sum_{k=1}^{n} p_k f(x_k) \geq \sum_{k=1}^{n} p_k f(y_k)$$

Proof:

We proceed by induction, for $n = 1$ the result is immediate. Now suppose that the theorem is true for every $n = 1,2, \cdots, m-1$ where m is a natural number, to complete the induction we wish to prove it for $n = m$. If $g(x_i) = g(y_i)$ for some index i, then by lemma 2.2.1 $f(x_i) = f(y_i)$, this would reduce the terms of the sum

[3] See [1] for reference.

$$\sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} p_k f(y_k)$$

to a sum with fewer terms, that is to say

$$\sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} p_k f(y_k) = \sum_{\substack{k=1 \\ k\neq i}}^{n} p_k f(x_k) - \sum_{\substack{k=1 \\ k\neq i}}^{n} p_k f(y_k)$$

then we can apply the induction hypothesis to conclude that

$$\sum_{\substack{k=1 \\ k\neq i}}^{n} p_k f(x_k) - \sum_{\substack{k=1 \\ k\neq i}}^{n} p_k f(y_k) \geq 0$$

and the theorem is proved for this case.

Suppose that $g(x_i) \neq g(y_i)$ for any i, then we can write

$$\sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} p_k f(y_k) = \sum_{k=1}^{n} p_k (f(x_k) - f(y_k))$$

$$= \sum_{k=1}^{n} p_k (g(x_k) - g(y_k)) \left(\frac{f(x_k) - f(y_k)}{g(x_k) - g(y_k)}\right)$$

We apply Abel summation formula as in Theorem 3.1.4 for $a_k = p_k(g(x_k) - g(y_k))$,

$b_k = \frac{f(x_k)-f(y_k)}{g(x_k)-g(y_k)}$, $p = 1$ and $q = n$ we conclude that the sum will equal to

$$\sum_{k=1}^{n-1} \left(\frac{f(x_k) - f(y_k)}{g(x_k) - g(y_k)} - \frac{f(x_{k+1}) - f(y_{k+1})}{g(x_{k+1}) - g(y_{k+1})}\right)\left(\sum_{i=1}^{k} p_i\big(g(x_i) - g(y_i)\big)\right)$$

$$+ \left(\sum_{i=1}^{n} p_i\big(g(x_i) - g(y_i)\big)\right)\left(\frac{f(x_n) - f(y_n)}{g(x_n) - g(y_n)}\right)$$

which is by condition iii) it reduces to

$$\sum_{k=1}^{n-1}\left(\frac{f(x_k)-f(y_k)}{g(x_k)-g(y_k)}-\frac{f(x_{k+1})-f(y_{k+1})}{g(x_{k+1})-g(y_{k+1})}\right)\left(\sum_{i=1}^{k}p_i\big(g(x_i)-g(y_i)\big)\right).$$

The proof will be complete if we can prove that the latest sum is nonnegative. Since by condition ii) we always have $\sum_{i=1}^{k}p_i\big(g(x_i)-g(y_i)\big)\geq 0$ then we only need to prove that

$$\frac{f(x_k)-f(y_k)}{g(x_k)-g(y_k)}\geq\frac{f(x_{k+1})-f(y_{k+1})}{g(x_{k+1})-g(y_{k+1})}.$$

If $g(x_k)=g(x_{k+1})$ for some index k then by lemma 2.2.1 $f(x_k)=f(x_{k+1})$, by condition i) we have $g(x_{k+1})\leq g(x_k)$. Thus by lemma 2.2.2

$$\frac{f(x_{k+1})-f(y_{k+1})}{g(x_{k+1})-g(y_{k+1})}=\frac{f(y_{k+1})-f(x_{k+1})}{g(y_{k+1})-g(x_{k+1})}$$

$$\leq\frac{f(y_k)-f(x_{k+1})}{g(y_k)-g(x_{k+1})}$$

$$=\frac{f(x_{k+1})-f(y_k)}{g(x_{k+1})-g(y_k)}$$

$$=\frac{f(x_k)-f(y_k)}{g(x_k)-g(y_k)}.$$

The case $g(y_k)=g(y_{k+1})$ can be handled similarly.

If $g(x_k)\neq g(x_{k+1})$ and $g(y_k)\neq g(y_{k+1})$ then by i) we have $g(x_k)>g(x_{k+1})$ and $g(y_k)>g(y_{k+1})$, we consider two cases :

<u>Case I</u>: $g(x_k)\neq g(y_{k+1})$. By lemma 2.2.2

$$\frac{f(x_k)-f(y_k)}{g(x_k)-g(y_k)} \geq \frac{f(x_{k+1})-f(y_k)}{g(x_{k+1})-g(y_k)}$$

$$= \frac{f(y_k)-f(x_{k+1})}{g(y_k)-g(x_{k+1})} \geq \frac{f(y_{k+1})-f(x_{k+1})}{g(y_{k+1})-g(x_{k+1})}$$

Case II: $g(x_k) = g(y_{k+1})$, then $g(y_k) > g(y_{k+1}) = g(x_k) > g(x_{k+1})$ and also by lemma 2.2.1 we have $f(x_k) = f(y_{k+1})$, so by lemma 2.2.2

$$\frac{f(x_k)-f(y_k)}{g(x_k)-g(y_k)} = \frac{f(y_k)-f(x_k)}{g(y_k)-g(x_k)} \geq \frac{f(x_{k+1})-f(x_k)}{g(x_{k+1})-g(x_k)} = \frac{f(x_{k+1})-f(y_{k+1})}{g(x_{k+1})-g(y_{k+1})}$$

Hence the theorem is proved.

Our final result is the Riemann Integral form of Theorem 3.2.1.

Theorem 3.2.2 (The Riemann Form of Generalization of Hardy-Littlewood- Pólya Majorization Inequality)

Let $f, g: X \to \mathbb{R}$ be two functions such that $g \lhd f$ and let $\phi, \varphi: [a,b] \to X$ are continuous functions such that $f \circ \phi$ and $f \circ \varphi$ are Riemann Integrable and $g \circ \phi$ and $g \circ \varphi$ are continuous nonincreasing functions. If $w: [a,b] \to \mathbb{R}$ is continuous function such that all two conditions below are satisfied:

i) $\int_a^x w(t)g(\phi(t))dt \geq \int_a^x w(t)g(\varphi(t))dt$ *for all* $x \in [a,b]$

ii) $\int_a^a w(t)g(\phi(t))dt = \int_a^b w(t)g(\varphi(t))dt$

then

$$\int_a^b w(t)f(\phi(t))dt \geq \int_a^b w(t)f(\varphi(t))dt$$

Proof:

Since $g \circ \phi$ and $g \circ \varphi$ are continuous and w is also continuous then by lemma 3.1.3 there exist sequences $c_k, d_k \in [x_{k-1}, x_k]$ where $x_k = a + \frac{(b-a)k}{n}$ such that

$$\big(w(c_1)(g \circ \phi)(c_1), \cdots, w(c_n)(g \circ \phi)(c_n)\big) \succ \big(w(d_1)(g \circ \varphi)(d_1), \cdots, w(d_n)(g \circ \varphi)(d_n)\big)$$

Thus by definition of Majorization we have:

i) $\sum_{i=1}^{m} w(c_i) g(\phi(c_i)) \geq \sum_{i=1}^{m} w(d_i) g(\varphi(d_i))$, for $m = 1,2,\cdots,n-1$

ii) $\sum_{i=1}^{n} w(c_i) g(\phi(c_i)) = \sum_{i=1}^{n} w(d_i) g(\varphi(d_i))$

Since we have $g \lhd f$, and $g \circ \phi$ and $g \circ \varphi$ are continuous, by Theorem 3.1.5

$$\sum_{k=1}^{n} w(c_k) f(\phi(c_k)) \geq \sum_{k=1}^{n} w(d_k) f(\varphi(d_k))$$

Each $c_k, d_k \in [x_{k-1}, x_k]$ with $x_k = a + \frac{(b-a)k}{n}$, and since $f \circ \phi$ and $f \circ \varphi$ are Riemann integrable and w is continuous, $w(t)(f \circ \phi)(t)$ and $w(t)(f \circ \varphi)(t)$ are Riemann Integrable, thus both limits $\lim_{n\to\infty} \sum_{k=1}^{n} w(c_k) f(\varphi(c_k)) \left(\frac{b-a}{n}\right)$ and $\lim_{n\to\infty} \sum_{k=1}^{n} w(d_k) f(\phi(d_k)) \left(\frac{b-a}{n}\right)$ exist and equal to their correspond integral and hence

$$\int_a^b w(t) f(\phi(t)) dt \geq \int_a^b w(t) f(\varphi(t)) dt \qquad \blacksquare.$$

3.3 Some Examples of Well-known Inequalities and nontrivial Inequalities

Now we will provide some applications of Theorem 3.1.2 and Theorem 3.2.1. Some examples provided here are already available in literatures usually with different proof.

Example 3.3.1 (The Arithmetic-Geometric Mean Inequality)

Let $x_1, x_2. \cdots, x_n$ *be nonnegative real numbers then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

Proof:

Without loss of generality, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, since when some x_i equal zero then the inequality is obvious. The array $(x_1, x_2, \cdots, x_n)$ majorizes $(y_1, y_2, \cdots, y_n)$ with $y_k = \frac{x_1+x_2+\cdots+x_n}{n}$. We use the Classical Hardy-Littlewood- Pólya Inequality with convex function $f(x) = -\ln x$ on $(0, \infty)$ and yields

$$-\ln x_1 - \ln x_2 - \cdots - \ln x_n \geq n\left[-\ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\right].$$

Which is equivalent to

$$\ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \geq \ln(x_1 x_2 \cdots x_n)^{\frac{1}{n}}.$$

This simplify to

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Example 3.3.2 (Power Mean Inequality)

Let b and a be positive real numbers such that $b \geq a$ then for any nonnegative real numbers $u_1, u_2, \cdots, u_n$ we have

$$\sqrt[b]{\frac{u_1^b + u_2^b + \cdots + u_n^b}{n}} \geq \sqrt[a]{\frac{u_1^a + u_2^a + \cdots + u_n^a}{n}}$$

Proof:

Without loss of generality we may assume that $u_1 \geq u_2 \geq \cdots \geq u_n$. Let $g(x) = x^a$ and $f(x) = x^b$ for all $x \in \mathbb{R}^+ \cup \{0\}$ where $b \geq a$ then by Definition 2.2.1 $g \lhd f$. Let $x_i = u_i^a$ and $y_i = \left[\frac{x_1+x_2+\cdots+x_n}{n}\right]$ then we have:

i) $g\left(x_1^{\frac{1}{a}}\right) \geq g\left(x_2^{\frac{1}{a}}\right) \geq \cdots \geq g\left(x_n^{\frac{1}{a}}\right)$

ii) $\sum_{i=1}^{m} g\left(x_i^{\frac{1}{a}}\right) \geq \sum_{i=1}^{m} g\left(y_i^{\frac{1}{a}}\right)$ for $m = 1,2,\cdots,n-1$

iii) $\sum_{i=1}^{n} g\left(x_i^{\frac{1}{a}}\right) = \sum_{i=1}^{n} g\left(y_i^{\frac{1}{a}}\right)$.

Thus by extended Hardy-Littlewood-Pólya Inequality we have

$$\frac{x_1^{b/a} + x_2^{b/a} + \cdots + x_n^{b/a}}{n} \geq \left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]^{\frac{b}{a}}$$

Let $x_i = u_i^a$ then we have

$$\sqrt[b]{\frac{u_1^b + u_2^b + \cdots + u_n^b}{n}} \geq \sqrt[a]{\frac{u_1^a + u_2^a + \cdots + u_n^a}{n}}$$

Example 3.3.3 (Jensen Inequality)

Let f be a real valued convex function defined in interval I, for any real numbers $\lambda_i \in [0,1]$ where $i = 1,2,\cdots,n$ satisfies $\sum_{i=1}^{n} \lambda_i = 1$ and $x_1, x_2, \cdots, x_n$ are all in I then

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$

Proof:

Without loss of generality we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$. For any positive rational numbers $r_1, r_2, \cdots, r_n$ such that $r_1 + r_2 + \cdots + r_n = 1$, let $r_i = \frac{p_i}{q_i}$ where $q_i, p_i \in \mathbb{N}$, and let $N = q_1 q_2 \cdots q_n$, then we have Nr_i is an positive integer for all i.

The array $(\underbrace{x_1, x_1, \cdots, x_1}_{Nr_1\ times}, \underbrace{x_2, x_2, \cdots x_2}_{Nr_2\ times}, \cdots, \underbrace{x_n, x_n, \cdots, x_n}_{Nr_n\ times})$ majorize the array $(y_1, y_2, \cdots, y_N)$ with $y_k = \frac{\sum_{i=1}^{n} Nr_i x_i}{N}$. Since f is convex, we can apply Classical Hardy-Littlewood- Pólya Inequality

$$\overbrace{\left(f(x_1) + \cdots + f(x_1)\right)}^{Nr_1\ times} + \overbrace{\left(f(x_2) + \cdots + f(x_2)\right)}^{Nr_2 times} + \cdots + \overbrace{\left(f(x_n) + \cdots + f(x_n)\right)}^{Nr_n times}$$

$$\geq Nf\left(\frac{(Nr_1)x_1 + (Nr_2)x_2 + \cdots + (Nr_n)x_n}{N}\right)$$

$$= Nf(r_1 x_1 + r_2 x_2 + \cdots + r_n x_n).$$

Which is equivalent to

$$Nr_1 f(x_1) + Nr_2 f(x_2) + \cdots + Nr_n f(x_n) \geq Nf(r_1 x_1 + r_2 x_2 + \cdots + r_n x_n).$$

Thus for any positive rational numbers r_i such that $r_1 + r_2 + \cdots + r_n = 1$ we have

$$r_1 f(x_1) + r_2 f(x_2) + \cdots + r_n f(x_n) \geq f(r_1 x_1 + r_2 x_2 + \cdots + r_n x_n)$$

It remains to prove the assertion for positive real numbers λ_i such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. For any real numbers λ_i there exist rational sequences $r_i^{(k)}$ that converges to λ_i and $r_1^{(k)} + r_2^{(k)} + \cdots + r_n^{(k)} = 1$. Applying the inequality for rational numbers we have

$$r_1^{(k)} f(x_1) + r_2^{(k)} f(x_2) + \cdots + r_n^{(k)} f(x_n) \geq f\left(r_1^{(k)} x_1 + r_2^{(k)} x_2 + \cdots + r_n^{(k)} x_n\right)$$

Taking limit as $k \to \infty$, since $\lim_{k\to\infty} r_i^{(k)} = \lambda_i$ and f is continuous we have

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$

The next example, although has a trivial geometric interpretation, but its analytic proof is not easy.

Example 3.3.4 (The Comparison of Arc Length)

Let $S: [a,b] \to \mathbb{R}$ and $T: [a,b] \to \mathbb{R}$ be a convex and continuously differentiable function such that $S(t) \leq T(t)$ for all $t \in [a,b]$, $S(a) = T(a) = 0$ and $S(b) = T(b)$, then at interval $[a,b]$ the arc length of S is less than or equal to the arc length of T.

Proof:

Since S and T is continuously differentiable and $S(a) = T(a) = 0$, then

$$S(t) = \int_a^t S'(x)dx \quad \text{and} \quad T(t) = \int_a^t T'(x)dx$$

The condition $S(t) \leq T(t)$ and $S(b) = T(b)$ is equivalent to

$$\int_a^t (-T'(x))dx \leq \int_a^t (-S'(x))dx \text{ and } \int_a^b (-S'(x)dx) = \int_a^b (-T'(x))dx.$$

This mean since S and T are convex then $-S'$ and $-T'$ are decreasing. Thus S and T satisfies the hypothesis of Riemann form of Classical Hardy-Littlewood- Pólya Inequality

By using the convex function $f(x) = \sqrt{1+x^2}$ in Riemann form of classical Hardy-Littlewood- Pólya Inequality we have

$$\int_a^b \sqrt{1 + \left(-S'(t)\right)^2}\, dt \leq \int_a^b \sqrt{1 + \left(-T'(t)\right)^2}\, dt$$

The left-hand side is the arc length of S in interval $[a, b]$ and the right-hand side is the arc length of T in interval $[a, b]$.