# CHAPTER II

# FUNDAMENTAL THEORIES

In this chapter, we will study about functions concerning with their convexity properties. We first begin with convex function which is the key role to state the Classical Hardy-Littlewood-Pólya Majorization Inequality. We will also give the definition of two functions that are relatively convex to each other. The definition of relative convexity given here is taken form **[1]**, but the first known study of relative convexity was due to George Pólya **[2].**

## 2.1 The Theories of Convex Functions

One of the elementary function in mathematics is convex functions, the definition of convex function merely states that any two points $(x_1, y_1)$ and $(x_2, y_2)$ on the curve of convex function must lie below the line joining the two given points. We give the formal definition below.

### Definition 2.1.1

Let $X \subset \mathbb{R}$ be a convex set. The function $f: X \to \mathbb{R}$ is *convex* if and only if for any two points $x_1, x_2 \in X$ and $\lambda \in [0,1]$ the condition

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \qquad (2.1.1)$$

is satisfied.

Throughout the following discussion, we will use the term "$f$ is a convex function" to refer that $f$ is a convex function defined on the subset $X$ of real number unless stated otherwise.

This typical class of functions has many interesting properties and often found in any branch of mathematics as well as in the operation research, mathematics for finance and science management. We shall derive some properties of convex function on this chapter for our further purpose that is to give a rigorous development of the famous inequalities due to G.H Hardy, J.E Littlewood and G.Pólya[1] and moreover its generalization.

By Definition 2.1.1, if $\lambda \neq \{0,1\}$ and $x_1 \neq x_2$ then the equality occurs only when $f$ is linear that is $f(x) = ax + b$ for constants $a$ and $b$. This observation is useful to determine the equality case of some inequalities involving convex function; this will be stated in the following theorem.

**Theorem 2.1.1**

*Let $x_1 < x_2$ be fixed real numbers, and $f$ is a convex function. for any $\lambda \in (0,1)$ the equality*

$$f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda)f(x_2)$$

*hold if and only if $f$ is a linear function on $(x_1, x_2)$.*

[1] Also founded independently by Karimata.

**Proof:**

First we prove that the equality implies $f$ is linear. For the sake of contradiction, suppose that $f$ is not linear in $(x_1, x_2)$ and for any $\lambda \in (0,1)$ we have

$$f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda)f(x_2).$$

For any point $x \in (x_1, x_2)$ we can write $x = \lambda x_1 + (1-\lambda)x_2$ where $\lambda = \frac{x_2 - x}{x_2 - x_1}$ and since $x \in (x_1, x_2)$ it follows that $\lambda \in (0,1)$. So for any $x \in (x_1, x_2)$ we have

$$\begin{aligned} f(x) &= f(\lambda x_1 + (1-\lambda)x_2) \\ &= \lambda f(x_1) + (1-\lambda)f(x_2) \\ &= \left(\frac{x_2 - x}{x_2 - x_1}\right) f(x_1) + \left(\frac{x - x_1}{x_2 - x_1}\right) f(x_2) \\ &= \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)(x - x_1) + f(x_1) \\ &= \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right) x + \left(\frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1}\right). \end{aligned}$$

But then $f$ has a form of $f(x) = ax + b$ with $a = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ and $b = \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1}$ that is $f$ a linear function. For the converse, suppose that $f$ is a linear function and has the form $f(x) = ax + b$ then for any $\lambda \in (0,1)$

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &= a(\lambda x_1 + (1-\lambda)x_2) + b \\ &= \lambda(a x_1 + b) + (1-\lambda)(a x_2 + b) \\ &= \lambda f(x_1) + (1-\lambda)f(x_2) \end{aligned}$$

and the theorem is established■.

Our next consideration is the property of convex functions concerning on the slope of the lines joining any two of the three given points on the convex function. The theorem that will be stated next comes from a geometric observation of the graph of convex function. Since it contributes many useful applications on deriving many properties of convex function, it is often referred as lemma.

**Theorem 2.1.2 (The Three Chords Lemma)**

*Let $f$ be a convex function and $x_1 < x_3 < x_2$ be any points on the domain of $f$ then we have*

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3} \qquad (2.1.2)$$

**Proof**:

Notice that $x_3 = \lambda x_1 + (1 - \lambda)x_2$ where $\lambda = \frac{x_2 - x_3}{x_2 - x_1}$ and from the condition $x_1 < x_3 < x_2$ we have $\lambda \in (0,1)$. By the definition of convex function we have

$$f(x_3) = f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

$$= \frac{x_2 - x_3}{x_2 - x_1} f(x_1) + \frac{x_3 - x_1}{x_2 - x_1} f(x_2)\,. \qquad (2.1.3)$$

By adding $-f(x_1)$ to the both sides of (2.1.3) we have

$$f(x_3) - f(x_1) \leq \frac{(x_3 - x_1)f(x_2) - (x_3 - x_1)f(x_1)}{x_2 - x_1}.$$

By dividing both sides with $x_3 - x_1 > 0$ we have

$$\frac{f(x_3)-f(x_1)}{x_3-x_1} \leq \frac{f(x_2)-f(x_1)}{x_2-x_1}. \qquad (2.1.4)$$

Hence if both sides of (2.1.3) are added with $-f(x_2)$ we have

$$f(x_3)-f(x_2) \leq \frac{(x_3-x_2)f(x_2)-(x_3-x_2)f(x_1)}{x_2-x_1}.$$

By dividing both sides with $x_3-x_2 \leq 0$ we have

$$\frac{f(x_3)-f(x_1)}{x_3-x_2} \geq \frac{f(x_2)-f(x_1)}{x_2-x_1}. \qquad (2.1.5)$$

The inequalities (2.1.4) and (2.1.5) give

$$\frac{f(x_3)-f(x_1)}{x_3-x_1} \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_2)-f(x_3)}{x_2-x_3}.$$ ■

Theorem 2.1.2 has a corollary, which will come into the consideration since it will be used frequently. This corollary states that convex functions possess an increased slope property.

**Corollary 2.1.1 (Galvani's Lemma)** *Let $f$ be a convex function on $[a,b]$ and $a \leq c < d \leq b$, then for any $x \in [a,b]/\{c,d\}$ the following inequality is true*

$$\frac{f(c)-f(x)}{c-x} \leq \frac{f(d)-f(x)}{d-x} \qquad (2.1.6)$$

*And if we have the equality in (2.1.6) then $f$ is linear.*

**Proof:**

Since $x \in [a,b]/\{c,d\}$, the proof will be constructed by examining three cases,

Case I. If $x < c < d$ then by the *three chords lemma* we have (the first inequality)

$$\frac{f(c)-f(x)}{c-x} \le \frac{f(d)-f(x)}{d-x}$$

Case II. If $c < x < d$ then by the *three chords lemma* we have (the second inequality)

$$\frac{f(c)-f(x)}{c-x} = \frac{f(x)-f(c)}{x-c} \le \frac{f(d)-f(x)}{d-x}$$

Case III. If $c < d < x$ then by the *three chords lemma* we have (the second inequality)

$$\frac{f(d)-f(x)}{d-x} = \frac{f(x)-f(d)}{x-d} \le \frac{f(x)-f(c)}{x-c} = \frac{f(c)-f(x)}{c-x}$$

thus the inequality has been proved. For the equality case, suppose that

$$\frac{f(d)-f(x)}{d-x} = \frac{f(c)-f(x)}{c-x}$$

for any $x \notin \{c,d\}$. Then we have $f(d)c - f(d)x - f(x)c = f(c)d - f(c)x - f(x)d$. This equivalent with

$$f(x) = \left(\frac{f(d)-f(c)}{d-c}\right)x + \left(\frac{f(c)d-f(d)c}{d-c}\right). \tag{2.1.7}$$

Equation 2.1.7 is also true for $x \in \{c,d\}$, and then $f$ is linear for any $x \in [a,b]$ ■.

The Galvani's lemma often rephrase as follow

**Corollary 2.1.2** *For any fixed* $t_0 \in [a,b]$ *the function*

$$\psi(t;t_0) = \frac{f(t)-f(t_0)}{t-t_0}, \quad t \neq t_0 \tag{2.1.8}$$

is nondecreasing when $t \in [a,b]$.

**Proof:**

To prove the rephrase version, let $\psi(t;t_0)$ be the function defined in 2.1.8. Let $t_1$ and $t_2$ be two points on $[a,b]$ which are different from $t_0$ and satisfy $a \le t_1 < t_2 \le b$, since $t_0 \in [a,b]/\{t_1,t_2\}$, by Galvani's lemma

$$\frac{f(t_1)-f(t_0)}{t_1-t_0} \le \frac{f(t_2)-f(t_0)}{t_2-t_0}$$

which is equivalent to $\psi(t_1;t_0) \le \psi(t_2;t_0)$ for any $t_1 < t_2$ ■ .

Although there are some discussions about discontinuous convex functions in **[2]**, we shall not discuss this class of convex function, it is irrelevant with the definition of convex function given here[2]. However, the convex function always continuous on closed interval as stated in the theorem below.

**Theorem 2.1.3**

*If $f$ is a convex function on $(a,b)$, then $f$ is continuous on each closed subinterval of $(a,b)$.*

**Proof:**

Let $[c,d] \subset (a,b)$, for any two different points $x,y \in (c,d)$ since $c \ne a$, $x \ne c$ and $a \le x$, by Galvani's lemma we have

$$\frac{f(a)-f(c)}{a-c} \le \frac{f(x)-f(c)}{x-c}$$

since $c \le y$ and $x \ne y$ we have by Galvani's lemma

[2] Hardy, Littlewood and Pólya gave different definition of convex function in **[2]**

$$\frac{f(x)-f(c)}{x-c} \leq \frac{f(x)-f(y)}{x-y}$$

since $x \leq b$ and $b \neq y$ by applying Galvani's lemma again we have

$$\frac{f(x)-f(y)}{x-y} \leq \frac{f(b)-f(y)}{b-y}$$

finally since $y \leq d$ and $b \neq d$ the Galvani's lemma gives

$$\frac{f(b)-f(y)}{b-y} = \frac{f(y)-f(b)}{y-b} \leq \frac{f(d)-f(b)}{d-b} = \frac{f(b)-f(d)}{b-d}$$

by the four previous inequalities, we have established the inequality

$$\frac{f(a)-f(c)}{a-c} \leq \frac{f(x)-f(y)}{x-y} \leq \frac{f(b)-f(d)}{b-d}.$$

Thus for any $x, y \in (c, d)$ with $x \neq y$ the slope $\frac{f(x)-f(y)}{x-y}$ is bounded when $x, y \in (c, d)$, if we allow $x, y \in [c, d]$ the slope will remain bounded there since $f(c)$ and $f(d)$ are defined. We conclude that the slope $\frac{f(x)-f(y)}{x-y}$ is bounded for $x, y \in [c, d]$ thus there exist real number $M > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq M \Longrightarrow |f(x)-f(y)| \leq M|x-y| \text{ for any } x, y \in [c, d]$$

or $|f(x)-f(y)| \leq M|x-y|$ for all $x, y \in [c, d]$. We have characterized $f$ as a Lipsitcz function and hence continuous ■.

**Theorem 2.1.4**

*If $f$ is a convex function on $(a, b)$, then the right- and left-hand derivatives of $f$ exist at each point of $(a, b)$.*

**Proof**:

By Galvani's lemma, for any $t_0 \in (a, b)$ the function $\psi(t; t_0) = \frac{f(t)-f(t_0)}{t-t_0}$ is nondecreasing on variable $t$, thus when $t \to t_0$ its left- and right-hand limit exist, finite and

$$\lim_{t \to t_0^+} \psi(t; t_0) = f'_+(t_0) \text{ and } \lim_{t \to t_0^-} \psi(t; t_0) = f'_-(t_0)$$

thus both $f'_+(t_0)$ and $f'_-(t_0)$ exist.

**Theorem 2.1.5**

*If $f$ is a convex function on $(a, b)$, then at each point of $(a, b)$ the right-hand derivatives is greater than or equal to left-hand derivatives of $f$.*

**proof:**

Let $t_0 \in (a, b)$, then $x < t_0 < y$ for some $x, y \in (a, b)$. As before, define $\psi(t; t_0) = \frac{f(t)-f(t_0)}{t-t_0}$, then by Galvani's lemma we have $\psi(x; t_0) \leq \psi(y; t_0)$. By theorem 2.1.4 the left-hand derivative exists, so when $x \to t_0^-$ (since $x < t_0$) we will have $f'_-(t_0) \leq \psi(y; t_0)$. Furthermore, the right-hand derivative exists, and when $y \to t_0^+$ (since $t_0 < y$) we will have $f'_-(t_0) \leq f'_+(t_0)$ as desired ■.

**Theorem 2.1.6**

*If $f$ is a convex function on $(a, b)$, then the left- and right-hand derivatives of $f$ are monotone increasing functions.*

**Proof**

Suppose that $t_1, t_2 \in (a, b)$ such that $t_1 < t_2$, then by Galvani's lemma for any $x$ such that $x < t_1 < t_2$ we have

$$\frac{f(x) - f(t_1)}{x - t_1} \leq \frac{f(x) - f(t_2)}{x - t_2}$$

Since as $x \to t_2^-$, we also have $x \to t_1^-$, the above inequality becomes $f_-^{'}(t_1) \leq f_-^{'}(t_2)$. Similarly, by the same method we have $f_+^{'}(t_1) \leq f_+^{'}(t_2)$, thus $f_-^{'}$ and $f_+^{'}$ are monotone functions ■.

**Theorem 2.1.7**

*If $f$ is a convex function on $(a, b)$, then at each point in $(a, b)$ the left-hand derivative of $f$ is less than or equal to its right-hand derivative and they are equal to each other except on a countable set.*

**Proof:**

Suppose that $h > 0$ and $t < x < t + h$ then by Galvani's lemma we have

$$\frac{f(x) - f(t+h)}{x - (t+h)} \geq \frac{f(t) - f(t+h)}{-h} = \frac{f(t+h) - f(t)}{h}. \qquad (2.1.7)$$

Letting $x \to (t+h)^-$ we have $f_-^{'}(t+h) \geq \frac{f(t+h)-f(t)}{h}$, if we suppose that $f_-^{'}$ to be continuous at $t$, then $\lim_{h\to 0^+} f_-^{'}(t+h) = f_-^{'}(t)$ and since $\lim_{h\to 0^+} \frac{f(t+h)-f(t)}{h} = f_+^{'}(t)$ we have

$$f_-^{'}(t) \geq f_+^{'}(t).$$

By Theorem 2.1.5 we also have $f_-^{'}(t) \leq f_+^{'}(t)$, so the equality $f_-^{'}(t) = f_+^{'}(t)$ holds if $f_-^{'}$ continuous on $t$. Since $f_-^{'}$ monotone, it can only have countable discontinuities, so the left- and hand-side derivatives are equal except on a countable set ■.

## 2.2 Relative Convexity

As our purpose is to generalize inequality concerning to convex functions, it should be quite natural to generalize the concept of convexity of the functions to relative convexity. The study of relative convexity can be trace back to George Pólya, and some of its properties are available in **[1]** which their proofs shall be reviewed again here. The motivation of relative convexity comes from the following observation,

**Theorem 2.2.1**

*The function $f$ is convex on $X$ if and only if for any $x_1, x_2, x_3 \in X$ and $x_1 \leq x_3 \leq x_2$ we have*

$$\begin{vmatrix} 1 & x_1 & f(x_1) \\ 1 & x_3 & f(x_3) \\ 1 & x_2 & f(x_2) \end{vmatrix} \geq 0 \qquad (2.2.1)$$

**Proof:**

If any two of $x_1, x_2$ and $x_3$ are equal to each other, then we have equality in (2.2.1) and the theorem is trivially proved. Suppose that $x_1, x_2$ and $x_3$ are different from each other, then by the three chords lemma since $x_1 < x_3 < x_2$ we have

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow (x_2 - x_1)f(x_3) + (x_3 - x_2)f(x_1) \leq (x_3 - x_1)f(x_2)$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3} \Rightarrow (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \leq (x_2 - x_3)f(x_1)$$

Adding up the two inequalities we have

$$(x_2 - x_3)f(x_1) + (x_3 - x_1)f(x_2) + (x_1 - x_2)f(x_3) \geq 0 \qquad (2.2.2)$$

That is equivalent to equation (2.2.1).

Now suppose that for any $x_1 \leq x_3 \leq x_2$ (2.2.1) holds, then (2.2.2) also holds. We can write $x_3 = \lambda x_1 + (1 - \lambda)x_2$ and substituting this to equation (2.2.2) yields

$$(x_2 - x_1)\lambda f(x_1) + (x_2 - x_1)(1 - \lambda)f(x_2) \geq (x_1 - x_2)f(\lambda x_1 + (1 - \lambda)x_2)$$

This is equivalent to equation (2.1.1). Thus $f$ is convex ■.

Theorem 2.2.1 motivates the definition of relative convexity that is to say that a function $f$ is convex with respect to the other nonconstant function of the same domain. This definition includes usual convexity as a special case by Theorem 2.2.1.

**Definition 2.2.1**

Let $f: X \to \mathbb{R}$ and let $g$ be a nonconstant function which has the same domain with $f$. We say that $f$ is convex with respect to $g$ (abbreviated, $g \lhd f$) if and only if

$$D(f,g) = \begin{vmatrix} 1 & g(x) & f(x) \\ 1 & g(y) & f(y) \\ 1 & g(z) & f(z) \end{vmatrix} \geq 0 \qquad (2.2.3)$$

For $x, y, z \in X$ and $g(x) \leq g(y) \leq g(z)$.

**Corollary 2.2.1**

*The function $f$ is convex if and only if $id \lhd f$ where $id$ is identity function.*

**Proof:**

Suppose that $f$ is convex, if $x_1 \leq x_3 \leq x_2$ then $id(x_1) \leq id(x_3) \leq id(x_2)$, hence by Theorem 2.2.1 $D(f, id) \geq 0$, thus we have $id \lhd f$. Now suppose that $id \lhd f$ then by definition $D(f, id) \geq 0$ for $x_1 \leq x_3 \leq x_2$ and hence by Theorem 2.2.1 $f$ convex.

**Lemma 2.2.1**

*Let $f, g: X \to \mathbb{R}$ are functions where $g$ is nonconstant such that $g \lhd f$, if $g(x) = g(y)$ then $f(x) = f(y)$.*

**Proof:**

Since $g$ is nonconstant, there exist $z \in X$ such that $g(x) = g(y) \neq g(z)$. Then we have two cases

<u>Case I:</u> $g(x) = g(y) < g(z)$ then

$$0 \le \begin{vmatrix} 1 & g(x) & f(x) \\ 1 & g(x) & f(y) \\ 1 & g(z) & f(z) \end{vmatrix} = \big(g(z) - g(x)\big)\big(f(x) - f(y)\big),$$

this yield $f(x) \ge f(y)$. And also we have

$$0 \le \begin{vmatrix} 1 & g(y) & f(y) \\ 1 & g(y) & f(x) \\ 1 & g(z) & f(z) \end{vmatrix} = \big(g(z) - g(y)\big)\big(f(y) - f(x)\big),$$

and this yields $f(y) \ge f(x)$, thus we have $f(x) = f(y)$.

Case II: $g(z) < g(x) = g(y)$ then

$$0 \le \begin{vmatrix} 1 & g(z) & f(z) \\ 1 & g(x) & f(x) \\ 1 & g(x) & f(y) \end{vmatrix} = \big(g(x) - g(z)\big)\big(f(y) - f(x)\big),$$

this yield $f(y) \ge f(x)$. And also we have

$$0 \le \begin{vmatrix} 1 & g(z) & f(z) \\ 1 & g(y) & f(y) \\ 1 & g(y) & f(x) \end{vmatrix} = \big(g(x) - g(z)\big)\big(f(x) - f(y)\big),$$

Which yields $f(x) \ge f(y)$, and hence $f(x) = f(y)$ ■.

The next lemma is a generalization of Galvani's lemma for relative convexity.

**Lemma 2.2.2**

If $g \lhd f$ then for any $a, u, v \in dom(f)$ with $g(u) \le g(v)$ and $g(a) \notin \{g(u), g(v)\}$, we have

$$\frac{f(u) - f(a)}{g(u) - g(a)} \le \frac{f(v) - f(a)}{g(v) - g(a)} \qquad (2.2.4)$$

**Proof:**

Notice that inequality (2.2.4) can written in the equivalent form

$$\big(g(u)-g(a)\big)\big(f(v)-f(a)\big)-\big(g(v)-g(a)\big)\big(f(u)-f(a)\big)\geq 0 \qquad (2.2.5)$$

To continue the proof we consider three cases:

<u>Case I</u>. If $g(a)<g(u)\leq g(v)$ then

$$0\leq\begin{vmatrix}1 & g(a) & f(a)\\ 1 & g(u) & f(u)\\ 1 & g(v) & f(v)\end{vmatrix}=\big(g(u)-g(a)\big)\big(f(v)-f(a)\big)-\big(g(v)-g(a)\big)\big(f(u)-f(a)\big)$$

which is equivalent to (2.2.5) and the inequality is verified for this case. Thus we have proved

$$\frac{f(u)-f(a)}{g(u)-g(a)}\leq\frac{f(v)-f(a)}{g(v)-g(a)}$$

for $g(a)<g(u)\leq g(v)$.

<u>Case II</u>**.** If $g(u)\leq g(v)<g(a)$ then

$$0\leq\begin{vmatrix}1 & g(u) & f(u)\\ 1 & g(v) & f(v)\\ 1 & g(a) & f(a)\end{vmatrix}=\big(g(u)-g(a)\big)\big(f(v)-f(a)\big)-\big(g(v)-g(a)\big)(f(u)-f(a))$$

which is (2.2.5) and the inequality is verified for this case. Thus we have proved

$$\frac{f(u)-f(a)}{g(u)-g(a)}\leq\frac{f(v)-f(a)}{g(v)-g(a)}$$

for $g(u)<g(v)\leq g(a)$.

Case III. If $g(u) < g(a) < g(v)$ then by Case I, for $g(u) < g(a) < g(v)$ we have

$$\frac{f(a) - f(u)}{g(a) - g(u)} \leq \frac{f(v) - f(u)}{g(v) - g(u)}$$

and then by Case II for $g(u) < g(a) < g(v)$ we have

$$\frac{f(u) - f(v)}{g(u) - g(v)} \leq \frac{f(a) - f(v)}{g(a) - g(v)}.$$

thus,

$$\frac{f(a) - f(u)}{g(a) - g(u)} \leq \frac{f(v) - f(u)}{g(v) - g(u)} = \frac{f(u) - f(v)}{g(u) - g(v)} \leq \frac{f(a) - f(v)}{g(a) - g(v)}$$

and the proof is complete.