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The Curvatures of Gradient Kähler-Ricci Solitons on the Complex Plane

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Preface

Why should it be math to be my skripsi topic?

Let's go back to about 3.5 years ago when I talked with Dr. Terry Mart in his office at the first time, he showed me that our current understanding about the universe ends up in the theory beyond Standard Model, and due to the lack of clue to have such a satisfying theory, physicists try to change the very fundamental assumption about universe, in which one of theories in this direction is the very popular yet controversial, superstring theory. He said that, to understand this theory fully, one must be expert in mathematics, especially the manipulation of space such as topology and differential geometry, because the central discussion in this theory is not far from it. He gave me an article about how we can change the shape of a cup into a doughnut and some touches about Poincaré conjecture which could make me very excited, not because the fact that this topic was in a peak period[1], but mostly because it is a really intriguing topic at least for some people who think in a geometrical way. But I must wait until I met Dr. Bobby Eka Gunara such that I could do my first real research about this matter[2]. Therefore it is very appropriate if I should mention their names as the ones who have great roles behind this work.

I also would thank to Dr. L.T. Handoko, Dr. Agus Salam, Dr. Imam Fachruddin, Dr. Anto Sulaksono and other (past-)members of Theoretical Physics Group of Universitas Indonesia (Chrisna S.N., M. Khalid Patmawijaya, Moch. Januar, Tjong Po Djun, Albertus Sulaiman, Muhandis

[1]Grisha Perelman got the Fields Medal (and declined it) due to his contribution to complete the Hamilton's technique in solving the Poincaré conjecture just one month before that afternoon talk with Sir Terry.

[2]and also my first serious research in high energy physics and math.

Shiddiq, Fathia R. Syahroni, Yunita Umniyati, Handhika S. Ramadhan, Suharyo Sumowidagdo, Andrias Fajarudin, Andhika Oxalion, Ryky Nelson, and Zuhrianda) for their supports and insanity to fill my days as a weird physics student in this university.

Talking about insanity, it does no longer make sense if I don't mention my friends in Physics UI '06 and also other friends in Physics '02, '03, '04, '05, '07, '08, '09, and all people in Math Department, who keep my spirit to be a good theoretical physicist (and a mathematician?). Sorry to disturb you with my unimportant behaviors and stormy laugh. For Physics Papa, Dr. Muhammad Hikam, I am very happy to be your teaching assistant in 3 years. Surely it is a very exciting experience to work and have fun with you. For my friends in Strawberry Home, who are ready to listen when I have freaky ideas, freaky songs, freaky smile, and who are ready when I am happy and when I am sad, I just feel that thanksgiving is not enough for all of you. For Manyang Panjerrino and Seanko Neri Anggi, thanks very much for your kindness. Sorry for all crazy things that messed up your days.

Not to be racist, I would like to thank my friends in Universitas Indonesia who have the same origin (and also the same senior high school), they are Andi Rosilala, Abdul Hadi Ilman, Shelli Eldita, Imam Jauhari, Kamal Hamzah, Farid Hosni (Physics '00), Benny Irawan (Physics '03), Desy Robiatul Adhawiyah (Physics '09), Ali Ihsanul Qauli (hopefully Physics '10 such that we will have a symmetry breaking), Danang Setyo Nugroho, and many others who accompanied me in migration from the beautiful island Madura into this land of the heathen, Depok. Let's dominate this university, guys.

For my colleagues in Tim Olimpiade Fisika Indonesia (TOFI) and Undergraduate Mathematics Competition, thanks for those awkward moments. For Arie Wibowo who has become my lecturer and my brother, thanks very much Kak. I wish I could write something on your board once more. For my closed friends Ajat Adriansyah and Lois Simandjuntak, in mathematical way of thinking, we believe.

Since I enjoy our universe through physics, math, and musics, I would say thanks to Coldplay, Oasis, Queen, U2, Mozart, Bach, Vivaldi, Beethoven,

Schubert, and M. Buble who always cheer up my feeling. Thanks for those great works. You are genius, buddies.

Beside those people who have many important roles in my life, I think it is my time to be back at home and say my great thanks for my parent, Abd. Latif and Nur Rahma, who undeniably give uncountable meanings in my life, and always support me with immeasurable patient. Although we have many difficult problems, I will always have a power to do my best for you. For my lovely sister and brother, Rika Nur Aftari Latief and Gammanda Adhny El-Zamzamy Latief, love the universe we live, do what you like and think what you want to think.

For Anggun Komala Sari, thanks for letting me into the AKS Universe; the dreamland for a weird theoretical physicist called *me*, in which all interactions are bundled into a single unified theory which is not only mathematically artistic, but also beautiful in a way that we could not ever think before. Thank you for the support you always give to me.

I wrote this work on behalf of the God Almighty, to love Him because His universe is indeed an oasis for the creatures who think.

Depok, April 16, 2010

Andy Octavian Latief

Abstract

By studying its curvatures, I prove the nontrivial equivalency between the constancy of Ricci scalar curvature and the Kähler-Einsteinian notion of gradient Kähler-Ricci solitons on the complex plane $\mathbb{C}^n$ in rotationally symmetric ansatz.

Keywords: Kähler-Ricci solitons, curvatures

Abstrak

Dengan mempelajari kelengkungannya, saya membuktikan ekivalensi antara nilai konstan kelengkungan skalar Ricci dan konsep Kähler-Einstein dari gradien Kähler-Ricci soliton pada bidang kompleks $\mathbb{C}^n$ dengan model simetri rotasi.

Kata kunci: Kähler-Ricci soliton, kelengkungan

Contents

...in a manner that will give the reader the clearest possible idea of *why* this theory takes the form it does, and why in this form it does such a good job of describing the real world.

Steven Weinberg

Chapter 1

Introduction

1.1 Background and Scope of Problem

In 1982 Richard Hamilton constructed the partial differential equation (PDE) which flows the Riemannian metric g_{ij} of a Riemannian manifold M along the negative direction of its Ricci curvature tensor R_{ij}, which mathematically has the form

$$\frac{\partial}{\partial t} g_{ij}(z,t) = -2R_{ij}(z,t), \qquad (z,t) \in M \times [0,\infty) \tag{1.1}$$

where now then called as *Ricci flow* [12]. By using this PDE, he hoped we can provide a method for proving the Thurston and Poincaré conjectures in classification of the Riemannian manifolds, of which the latter is one of the seven Millenium Prize problems need to be solved by the mathematicians for this and next centuries. But the method was still unable to prove these conjectures until Grigori Perelman made a series of three papers which finished Hamilton's program to solve them [17, 18, 19]. Some authors also have made a complete verification about the Perelman's work [5, 14], and all is agree that these conjectures are officially solved.

The Ricci flow can also be considered as a flow of the Kähler metric $g_{i\bar{j}}$ in Kähler manifold M along the negative direction of its Ricci curvature tensor $R_{i\bar{j}}$, or

$$\frac{\partial}{\partial t} g_{i\bar{j}}(z,t) = -R_{i\bar{j}}(z,t), \qquad (z,t) \in M \times [0,\infty) \tag{1.2}$$

where the omission of factor 2 in the RHS of equation (1.2) above is merely a matter of convention. It is known that for every $t \in [0,\infty)$ the metric $g_{i\bar{j}}$

is still Kähler, and hence this PDE is called the *Kähler-Ricci flow*. Although the existence and uniqueness of the solution of equation (1.1) are not obvious because it is a weakly parabolic system, finally it had been proved by some authors [12, 8, 7]. In fact, the system of Ricci flow is similar with the curve-shortening flow [20], which is the weakly parabolic PDE that has a simpler intuition than behavior of the Ricci flow itself. However, for the equation (1.2), it turns out that it is the strongly parabolic system, and hence the solution exists and it is unique [1]. The problem of convergence of this solution is also attacked by some authors [4, 6].

If the solution of equation (1.2) evolves under a certain one-parameter family of biholomorphism, or explicitly,

$$g_{i\bar{j}}(z,t) = \sigma(t)\varphi_t^* g_{i\bar{j}}(z,0) \tag{1.3}$$

where $\sigma(t)$ is the scaling function and φ_t^* is the induced mapping of the one-parameter family of biholomorphism φ_t, then the solution is called *Kähler-Ricci soliton*. Furthermore, if the vector which generates that biholomorphism is a gradient of some real-valued function, then we call this solution as the *gradient Kähler-Ricci soliton*.

The existence of gradient Kähler-Ricci soliton in a certain Kähler manifold had been showed by some authors in the case where the manifold is n-dimensional complex plane $\mathbb{C}^n$ and the soliton is restricted to be rotationally invariant [2, 3, 10, 11]. In [2], it was also proved that this soliton has positive sectional curvature. However, the behavior of this rotationally invariant gradient Kähler-Ricci soliton in $\mathbb{C}^n$ is not completely understood yet, and there are numerous structures of curvatures of this soliton remain unclear until today.

1.2 Research Aim

This research has a main aim to reveal the structure of curvatures of the rotationally invariant gradient Kähler-Ricci soliton in $\mathbb{C}^n$, especially the Ricci curvature tensor and Ricci scalar curvature, and study the relation between them.

1.3 Research Method

I use the general formulations for curvatures of Kähler metric and use them to attack the description of curvatures for rotationally invariant gradient Kähler-Ricci soliton in $\mathbb{C}^n$ such that we can study the structure of these curvatures.

Chapter 2

The Concept of Complex Geometry

In this chapter we discuss much about the structure of complex geometry. Any readers who don't have some preliminaries in differential geometry should read appendix B and C first before tackling this chapter.

2.1 Definition of Complex Manifolds

Complex manifold is defined roughly as the geometrical object which locally has structure similar with the complex Euclidean plane. It is common if we have a geometrical space then we should construct the coordinate system to label the points of this manifold with n-tuples of numbers, to make any calculation easier in this space. We are very familiar with this matter in the case of flat Euclidean space $\mathbb{C}^n$, in which the coordinate system is denoted as $\{z^\mu\}$ for $1 \leq \mu \leq n$, for a natural number n. For a complex manifold M, we can attach a grid of coordinate system such that it covers some parts of M, hence we can label the points of M by the values of coordinates which coincide with those points. This is very common and simple technical way to introduce the coordinate system in a manifold, yet it is very hard to formalize the concept of coordinate system. We should make a different point of view to construct this coordinate system in M.

Take a point $p \in M$ and construct the *neighborhood*[1] $U \subset M$ such that

[1] It is the very definition of manifold (not only the complex one) that we are always able to construct the neighborhood around a point. See appendix B.2.

U contains p. Then the coordinate system in a manifold can be viewed as the mapping from the points in a neighborhood U to the points in complex Euclidean plane $\mathbb{C}^n$. Define this mapping to be ϕ, then

$$\phi : U \to \mathbb{C}^n : p \mapsto \{z^\mu\}, \qquad \text{for } \mu = \{1, 2, \ldots, n\} \tag{2.1}$$

It is important to note that the role of ϕ, besides providing the coordinate system for the local neighborhoods of a manifold, is to manifest the local similarity with the complex Euclidean plane.

An interesting situation happens when we have two neighborhoods U and V which contain a point $p \in M$. If the coordinate mapping in neighborhood U is $\phi : U \to \mathbb{C}^n$, and coordinate mapping in neighborhood V is $\psi : V \to \mathbb{C}^n$, then the point p has two different coordinates $\phi(p)$ and $\psi(p)$. The mapping $\chi : \phi(U) \to \psi(V)$ then is a mapping from the subset of $\mathbb{C}^n$ to itself. We restrict, for a space M to be a complex manifold, that this mapping χ, for every neighborhoods $U, V \subset M$, must be the *holomorphic map*[2].

Since the complex manifold M is locally like the complex Euclidean plane $\mathbb{C}^n$, then the *dimension* of M is defined as the dimension of $\mathbb{C}^n$. However, the space $\mathbb{C}^n$ itself can be identified as the real Euclidean plane $\mathbb{R}^{2n}$ with dimension $2n$, then we should distinguish two types of dimension of a complex manifold M, the *complex dimension* $\dim_{\mathbb{C}} M$ and *real dimension* $\dim_{\mathbb{R}} M$, such that if $\dim_{\mathbb{C}} M = n$, then $\dim_{\mathbb{R}} M = 2n$.

2.2 Calculus on Complex Manifolds

2.2.1 Holomorphic Map

A function $f : \mathbb{C}^n \to \mathbb{C}$, where $f = u + iv$, is called *holomorphic* if for the coordinate $z^\mu = x^\mu + iy^\mu$ these relations hold

$$\frac{\partial u}{\partial x^\mu} = \frac{\partial v}{\partial y^\mu}, \qquad \frac{\partial u}{\partial y^\mu} = -\frac{\partial v}{\partial x^\mu} \tag{2.2}$$

for each μ, where $1 \leq \mu \leq n$. Similarly, the mapping $(f_1, \ldots, f_n) : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic if each f_α is holomorphic, for $1 \leq \alpha \leq n$.

[2]See subsection 2.2.1.

If we have a mapping $f : M \to N$, where M and N are complex manifolds, then f is holomorphic if for each point p in M and neighborhood $U \subset M$ which contains p, then $g : \phi(U) \to \psi(V)$ is holomorphic, where $V \subset N$ is the neighborhood of $f(p)$ in N, ϕ and ψ are the coordinate mappings in U and V.

2.2.2 Complexification

Since the concept of vector field in manifold is indispensable (appendix B.5), we should learn how to construct the vector field in complex manifold. If we have a vector field V, then the *complexification* of V is the set of vectors $A+iB$, where $A, B \in V$. The complexification of the vector field V is denoted as $V^{\mathbb{C}}$. In this way, we can complexify the tangent vector field T_pM, tangent dual vector field T_p^*M such that they will be $T_pM^{\mathbb{C}}$ and $T_p^*M^{\mathbb{C}}$.

The dimension of $V^{\mathbb{C}}$ is the same as V, the fact which easily can be seen from our construction of vectors in $V^{\mathbb{C}}$. If basis vectors of V are $\{e_\mu\} = \{e_1, \ldots, e_n\}$ then a vector $A + iB \in V^{\mathbb{C}}$, for $A, B \in V$, can be stated as

$$A + iB = (A^\mu + iB^\mu)e_\mu \tag{2.3}$$

and it implies that $\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V$.

2.2.3 Almost Complex Structure

If a manifold M is equipped with vector fields, then it must have the basis vectors. If we consider M as a real manifold with $\dim_{\mathbb{R}} M = 2n$, then the basis vectors are

$$\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\} \tag{2.4}$$

where the coordinates in M are $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$, and the basis dual vectors are

$$\{dx^1, \ldots, dx^n, dy^1, \ldots, dy^n\} \tag{2.5}$$

We can form the basis vectors for complex manifold M by defining

$$\frac{\partial}{\partial z^\mu} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\Big) \tag{2.6}$$

$$\frac{\partial}{\partial \bar{z}^\mu} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\Big) \tag{2.7}$$

and for the dual vectors,

$$
\begin{aligned}
dz^\mu &\equiv dx^\mu + i dy^\mu && (2.8)\\
d\bar{z}^\mu &\equiv dx^\mu - i dy^\mu && (2.9)
\end{aligned}
$$

where $1 \le \mu \le n$. Note that these basis vectors and dual vectors satisfy the orthonormality condition

$$
\Big\{dz^\mu, \frac{\partial}{\partial \bar{z}^\nu}\Big\} = \Big\{d\bar{z}^\mu, \frac{\partial}{\partial z^\nu}\Big\} = 0, \qquad \Big\{dz^\mu, \frac{\partial}{\partial z^\nu}\Big\} = \Big\{dz^\mu, \frac{\partial}{\partial z^\nu}\Big\} = \delta^\mu_\nu \quad (2.10)
$$

Then we can use the basis vectors (2.6) and (2.7) and the basis dual vectors (2.8) and (2.9) for complex manifold M.

Back to our view of M as a real manifold, we can define a mapping which maps a basis vector to another, $J : T_pM \to T_pM$, such that

$$
J\Big(\frac{\partial}{\partial x^\mu}\Big) = \frac{\partial}{\partial y^\mu}, \qquad J\Big(\frac{\partial}{\partial y^\mu}\Big) = -\frac{\partial}{\partial x^\mu} \quad (2.11)
$$

for every μ. This function is called *almost complex structure* since J^2 is minus the identity in T_pM. Indeed, the almost complex structure J is a real tensor of type $(1, 1)$, where in the explicit form it is

$$
J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad (2.12)
$$

with I is the $n \times n$ identity matrix.

Since the tangent vector space T_pM can be complexified into $T_pM^{\mathbb{C}}$, then the almost complex structure J must be generalized to yield

$$
J\Big(\frac{\partial}{\partial z^\mu}\Big) = i\frac{\partial}{\partial z^\mu}, \qquad J\Big(\frac{\partial}{\partial \bar{z}^\mu}\Big) = -\frac{\partial}{\partial \bar{z}^\mu} \quad (2.13)
$$

where two equations above come from the corresponding action of J to the basis vectors $\partial/\partial x^\mu$ and $\partial/\partial y^\mu$. Hence the explicit expression for the almost complex structure is

$$
J = i\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i\, d\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu} \quad (2.14)
$$

From these equations we can easily observe that the tangent vector fields in a complex manifold M can be decomposed as

$$
T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^- \quad (2.15)
$$

where

$$T_pM^{\pm} = \{Z \in T_pM^{\mathbb{C}} \mid JZ = \pm iZ\} \tag{2.16}$$

Another interesting fact is about the dimension of each spaces above. Since M has complex dimension n and real dimension $2n$, then the complex dimension of $T_pM^{\mathbb{C}}$ is $2n$, because it comes from the complexification of T_pM and we know that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$. Therefore we have

$$\begin{aligned} n &= \dim_{\mathbb{C}} M = \frac{1}{2}\dim_{\mathbb{R}} M && (2.17)\\ &= \frac{1}{2}\dim_{\mathbb{C}} T_pM^{\mathbb{C}} = \dim_{\mathbb{C}} T_pM^{+} = \dim_{\mathbb{C}} T_pM^{-} && (2.18)\end{aligned}$$

2.3 Hermitian Manifolds

Given a complex manifold M with $\dim_{\mathbb{C}} M = n$ equipped with a Riemannian metric g, then for two vectors $W = U + iV, Z = X + iY \in T_pM^{\mathbb{C}}$ we can extend g such that

$$g(W, Z) = g(U, X) - g(V, Y) + i[g(U, Y) + g(V, X)] \tag{2.19}$$

for every $U, V, X, Y \in T_pM$. Then the components of g are

$$\begin{aligned} g_{\mu\nu} &= g\Big(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial z^{\nu}}\Big) && (2.20)\\ g_{\mu\bar{\nu}} &= g\Big(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial \bar{z}^{\nu}}\Big) && (2.21)\\ g_{\bar{\mu}\nu} &= g\Big(\frac{\partial}{\partial \bar{z}^{\mu}}, \frac{\partial}{\partial z^{\nu}}\Big) && (2.22)\\ g_{\bar{\mu}\bar{\nu}} &= g\Big(\frac{\partial}{\partial \bar{z}^{\mu}}, \frac{\partial}{\partial \bar{z}^{\nu}}\Big) && (2.23)\end{aligned}$$

2.3.1 Hermitian Metric

If the Riemannian metric g of M satisfies the condition

$$g(JX, JY) = g(X, Y) \tag{2.24}$$

for any $X, Y \in T_pM$ and the point $p \in M$, then g is called the *Hermitian metric*. The manifold M which is equipped by Hermitian metric g is called *Hermitian manifold*. It is a well-known theorem which states that any complex manifold admits Hermitian metric. For a complex manifold M with any Riemannian metric g, then we can construct another metric g' which is defined as

$$g'(X,Y) = \frac{1}{2}\Big(g(X,Y) + g(JX,JY)\Big) \tag{2.25}$$

and it is obvious that $g'(JX,JY) = g'(X,Y)$, which implies that it is indeed a Hermitian metric.

By using the condition (2.24) for the Hermitian metric, we will know that the only components which are nonzero in this metric are the mixed components. The zero components are

$$g_{\mu\nu} = g\Big(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\Big) = g\Big(J\frac{\partial}{\partial z^\mu}, J\frac{\partial}{\partial z^\nu}\Big) = -g_{\mu\nu} = 0 \tag{2.26}$$

$$g_{\bar{\mu}\bar{\nu}} = g\Big(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\Big) = g\Big(J\frac{\partial}{\partial \bar{z}^\mu}, J\frac{\partial}{\partial \bar{z}^\nu}\Big) = -g_{\bar{\mu}\bar{\nu}} = 0 \tag{2.27}$$

Therefore, the Hermitian metric g can be written as

$$g = g_{\mu\bar{\nu}}\, dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\nu}\, d\bar{z}^\mu \otimes dz^\nu \tag{2.28}$$

2.3.2 Kähler Form

For a Hermitian manifold M with a Hermitian metric g, and for any $X, Y \in T_pM$, we can define a tensor

$$\Omega(X,Y) = g(JX,Y) \tag{2.29}$$

which then is called the *Kähler form* of Hermitian metric g. The Kähler form is antisymmetric, in a sense that

$$\begin{aligned} \Omega(X,Y) &= g(JX,Y) = g(J^2X,JY) = g(-X,JY) = -g(JY,X) \\ &= -\Omega(Y,X) \end{aligned} \tag{2.30}$$

If we see M as a complex manifold such that we should talk about the Kähler form in the domain of complex tangent vector space $T_pM^{\mathbb{C}}$, then the components of Kähler form Ω are zero except the mixed ones,

$$\Omega_{\mu\nu} = \Omega\Big(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\Big) = g\Big(J\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\Big) = ig_{\mu\nu} = 0 \tag{2.31}$$

$$\Omega_{\bar{\mu}\bar{\nu}} = \Omega\Big(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\Big) = g\Big(J\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\Big) = -ig_{\bar{\mu}\bar{\nu}} = 0 \tag{2.32}$$

$$\Omega_{\mu\bar{\nu}} = \Omega\Big(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\Big) = g\Big(J\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\Big) = ig_{\mu\bar{\nu}} \tag{2.33}$$

$$\Omega_{\bar{\mu}\nu} = \Omega\Big(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}\Big) = g\Big(J\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}\Big) = -ig_{\bar{\mu}\nu} \tag{2.34}$$

Therefore the Kähler form can be expressed as

$$\begin{aligned}
\Omega &= ig_{\mu\bar{\nu}}dz^{\mu} \otimes d\bar{z}^{\nu} - ig_{\bar{\mu}\nu}d\bar{z}^{\mu} \otimes dz^{\nu} && (2.35)\\
&= ig_{\mu\bar{\nu}}dz^{\mu} \otimes d\bar{z}^{\nu} - ig_{\mu\bar{\nu}}d\bar{z}^{\nu} \otimes dz^{\mu} && (2.36)\\
&= ig_{\mu\bar{\nu}}\, dz^{\mu} \wedge d\bar{z}^{\nu} && (2.37)
\end{aligned}$$

where the last term in the RHS is defined by

$$dz^{\mu} \wedge d\bar{z}^{\nu} \equiv dz^{\mu} \otimes d\bar{z}^{\nu} - d\bar{z}^{\nu} \otimes dz^{\mu} \tag{2.38}$$

2.4 Kähler Manifolds

2.4.1 Definition

If the Hermitian manifold M with a Hermitian metric g has a closed Kähler form,

$$d\Omega = 0 \tag{2.39}$$

then M is called *Kähler manifold* and g is called *Kähler metric*.

By using this condition, we will have

$$\begin{aligned}
d\Omega &= (\partial + \bar{\partial})ig_{\mu\bar{\nu}}dz^{\mu} \wedge d\bar{z}^{\nu}\\
&= i\partial_{\lambda}g_{\mu\bar{\nu}}dz^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\nu} + i\partial_{\bar{\lambda}}g_{\mu\bar{\nu}}d\bar{z}^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\nu}\\
0 &= \frac{1}{2}i(\partial_{\lambda}g_{\mu\bar{\nu}} - \partial_{\mu}g_{\lambda\bar{\nu}})dz^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\nu}\\
&\quad + \frac{1}{2}i(\partial_{\bar{\lambda}}g_{\mu\bar{\nu}} - \partial_{\bar{\nu}}g_{\mu\bar{\lambda}})d\bar{z}^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\nu} && (2.40)
\end{aligned}$$

and it implies that there are two equations that must be satisfied by Kähler metric,

$$\partial_{\lambda}g_{\mu\bar{\nu}} = \partial_{\mu}g_{\lambda\bar{\nu}}, \qquad \partial_{\bar{\lambda}}g_{\mu\bar{\nu}} = \partial_{\bar{\nu}}g_{\mu\bar{\lambda}} \tag{2.41}$$

The solution for these equations is the Kähler metric of the form

$$g_{\mu\bar{\nu}} = \partial_{\mu}\partial_{\bar{\nu}}K \tag{2.42}$$

where a scalar function K is called the *Kähler potential*.

2.4.2 The Curvatures of Kähler Manifolds

We are now in a position to talk about the curvatures of Kähler manifolds which will be our main discussion in the next chapter. The readers who are not familiar to this topic are very recommended to see appendix C.4.

Riemann Curvature Tensor

Generally the *Riemann curvature tensor* of the Kähler metric is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.43}$$

where X, Y and Z are the vector fields in $\mathbb{C}^n$. We can obtain the components of this Riemann tensor by setting each vector fields X, Y, Z to be the holomorphic and anti-holomorphic vector fields.

The most simple components are $R^{\lambda}{}_{\mu\nu\sigma}$ and $R^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}\bar{\sigma}}$, which are zero.

$$\begin{aligned}
R^{\lambda}{}_{\mu\nu\sigma} e_{\lambda} &= \nabla_{\nu}\nabla_{\sigma} e_{\mu} - \nabla_{\sigma}\nabla_{\nu} e_{\mu} - \nabla_{[e_{\nu},e_{\sigma}]} e_{\mu} \\
&= \nabla_{\nu}(\Gamma^{\rho}{}_{\sigma\mu} e_{\rho}) - \nabla_{\sigma}(\Gamma^{\rho}{}_{\nu\mu} e_{\rho}) \\
&= \nabla_{\nu} e_{\rho} \Gamma^{\rho}{}_{\sigma\mu} + \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} e_{\rho} - \nabla_{\sigma} e_{\rho}\Gamma^{\rho}{}_{\nu\mu} - \partial_{\sigma}\Gamma^{\rho}{}_{\nu\mu} e_{\rho} \\
&= \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} e_{\rho} - \partial_{\sigma}\Gamma^{\rho}{}_{\nu\mu} e_{\rho} + \Gamma^{\rho}{}_{\sigma\mu}\Gamma^{\omega}{}_{\nu\rho} e_{\omega} - \Gamma^{\rho}{}_{\nu\mu}\Gamma^{\omega}{}_{\sigma\rho} e_{\omega} \\
&= (\partial_{\nu}\Gamma^{\lambda}{}_{\sigma\mu} - \partial_{\sigma}\Gamma^{\lambda}{}_{\nu\mu} + \Gamma^{\rho}{}_{\sigma\mu}\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\rho}{}_{\nu\mu}\Gamma^{\lambda}{}_{\sigma\rho}) e_{\lambda}
\end{aligned}$$

such that we have

$$\begin{aligned}
R^{\lambda}{}_{\mu\nu\sigma} &= \partial_{\nu}\Gamma^{\lambda}{}_{\sigma\mu} - \partial_{\sigma}\Gamma^{\lambda}{}_{\nu\mu} + \Gamma^{\rho}{}_{\sigma\mu}\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\rho}{}_{\nu\mu}\Gamma^{\lambda}{}_{\sigma\rho} \\
&= \partial_{\nu}(g^{\lambda\bar{\omega}}\partial_{\sigma} g_{\mu\bar{\omega}}) - \partial_{\sigma}(g^{\lambda\bar{\omega}}\partial_{\nu} g_{\mu\bar{\omega}}) \\
&\quad +(g^{\rho\bar{\omega}}\partial_{\sigma} g_{\mu\bar{\omega}})(g^{\lambda\bar{\eta}}\partial_{\nu} g_{\rho\bar{\eta}}) - (g^{\rho\bar{\omega}}\partial_{\nu} g_{\mu\bar{\omega}})(g^{\lambda\bar{\eta}}\partial_{\sigma} g_{\rho\bar{\eta}}) \\
&= \partial_{\nu} g^{\lambda\bar{\omega}}\partial_{\sigma} g_{\mu\bar{\omega}} + g^{\lambda\bar{\omega}}\partial_{\nu}\partial_{\sigma} g_{\mu\bar{\omega}} - \partial_{\sigma} g^{\lambda\bar{\omega}}\partial_{\nu} g_{\mu\bar{\omega}} - g^{\lambda\bar{\omega}}\partial_{\sigma}\partial_{\nu} g_{\mu\bar{\omega}} \\
&\quad +g^{\rho\bar{\omega}} g^{\lambda\bar{\eta}}\partial_{\sigma} g_{\mu\bar{\omega}}\partial_{\nu} g_{\rho\bar{\eta}} - g^{\rho\bar{\omega}} g^{\lambda\bar{\eta}}\partial_{\nu} g_{\mu\bar{\omega}}\partial_{\sigma} g_{\rho\bar{\eta}} \\
&= \partial_{\nu} g^{\lambda\bar{\omega}}\partial_{\sigma} g_{\mu\bar{\omega}} - \partial_{\sigma} g^{\lambda\bar{\omega}}\partial_{\nu} g_{\mu\bar{\omega}} - \partial_{\sigma} g_{\mu\bar{\omega}}\partial_{\nu} g^{\lambda\bar{\omega}} + \partial_{\nu} g_{\mu\bar{\omega}}\partial_{\sigma} g^{\lambda\bar{\omega}} \\
&= 0
\end{aligned} \tag{2.44}$$

And similarly for $R^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}\bar{\sigma}}$, since it is the conjugate of $R^{\lambda}{}_{\mu\nu\sigma}$.

We also can verify that all other components except $R^{\lambda}{}_{\mu\bar{\nu}\sigma}$, $R^{\lambda}{}_{\mu\nu\bar{\sigma}}$, $R^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}\sigma}$ and $R^{\bar{\lambda}}{}_{\bar{\mu}\nu\bar{\sigma}}$ are zero. Here we have

$$
\begin{aligned}
R^{\lambda}{}_{\mu\bar{\nu}\sigma}e_{\lambda} &= \nabla_{\bar{\nu}}\nabla_{\sigma}e_{\mu} - \nabla_{\sigma}\nabla_{\bar{\nu}}e_{\mu} - \nabla_{[e_{\bar{\nu}},e_{\sigma}]}e_{\mu} \\
&= \nabla_{\bar{\nu}}(\Gamma^{\rho}{}_{\sigma\mu}e_{\rho}) = \partial_{\bar{\nu}}\Gamma^{\rho}{}_{\sigma\mu}e_{\rho} \qquad (2.45) \\
R^{\lambda}{}_{\mu\nu\bar{\sigma}}e_{\lambda} &= \nabla_{\nu}\nabla_{\bar{\sigma}}e_{\mu} - \nabla_{\bar{\sigma}}\nabla_{\nu}e_{\mu} - \nabla_{[e_{\nu},e_{\bar{\sigma}}]}e_{\mu} \\
&= -\nabla_{\bar{\sigma}}(\Gamma^{\rho}{}_{\nu\mu}e_{\rho}) = -\partial_{\bar{\sigma}}\Gamma^{\rho}{}_{\nu\mu}e_{\rho} \qquad (2.46) \\
R^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}\sigma}e_{\bar{\lambda}} &= \nabla_{\bar{\nu}}\nabla_{\sigma}e_{\bar{\mu}} - \nabla_{\sigma}\nabla_{\bar{\nu}}e_{\bar{\mu}} - \nabla_{[e_{\bar{\nu}},e_{\sigma}]}e_{\bar{\mu}} \\
&= -\nabla_{\sigma}(\Gamma^{\bar{\rho}}{}_{\bar{\nu}\bar{\mu}}e_{\bar{\rho}}) = -\partial_{\sigma}\Gamma^{\bar{\rho}}{}_{\bar{\nu}\bar{\mu}}e_{\bar{\rho}} \qquad (2.47) \\
R^{\bar{\lambda}}{}_{\bar{\mu}\nu\bar{\sigma}}e_{\bar{\lambda}} &= \nabla_{\nu}\nabla_{\bar{\sigma}}e_{\bar{\mu}} - \nabla_{\bar{\sigma}}\nabla_{\nu}e_{\bar{\mu}} - \nabla_{[e_{\nu},e_{\bar{\sigma}}]}e_{\bar{\mu}} \\
&= \nabla_{\nu}(\Gamma^{\bar{\rho}}{}_{\bar{\sigma}\bar{\mu}}e_{\bar{\rho}}) = \partial_{\nu}\Gamma^{\bar{\rho}}{}_{\bar{\sigma}\bar{\mu}}e_{\bar{\rho}} \qquad (2.48)
\end{aligned}
$$

such that it implies

$$
\begin{aligned}
R^{\lambda}{}_{\mu\bar{\nu}\sigma} &= \partial_{\bar{\nu}}\Gamma^{\lambda}{}_{\sigma\mu} = \partial_{\bar{\nu}}(g^{\lambda\bar{\rho}}\partial_{\sigma}g_{\mu\bar{\rho}}) \\
&= g^{\lambda\bar{\rho}}\partial_{\bar{\nu}}\partial_{\sigma}g_{\mu\bar{\rho}} + \partial_{\bar{\nu}}g^{\lambda\bar{\rho}}\partial_{\sigma}g_{\mu\bar{\rho}} \qquad (2.49) \\
R^{\lambda}{}_{\mu\nu\bar{\sigma}} &= -\partial_{\bar{\sigma}}\Gamma^{\lambda}{}_{\nu\mu} = -\partial_{\bar{\sigma}}(g^{\lambda\bar{\rho}}\partial_{\nu}g_{\mu\bar{\rho}}) \\
&= -g^{\lambda\bar{\rho}}\partial_{\bar{\sigma}}\partial_{\nu}g_{\mu\bar{\rho}} - \partial_{\bar{\sigma}}g^{\lambda\bar{\rho}}\partial_{\nu}g_{\mu\bar{\rho}} \qquad (2.50) \\
R^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}\sigma} &= -\partial_{\sigma}\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} = -\partial_{\sigma}(g^{\rho\bar{\lambda}}\partial_{\bar{\nu}}g_{\rho\bar{\mu}}) \\
&= -g^{\rho\bar{\lambda}}\partial_{\sigma}\partial_{\bar{\nu}}g_{\rho\bar{\mu}} - \partial_{\sigma}g^{\rho\bar{\lambda}}\partial_{\bar{\nu}}g_{\rho\bar{\mu}} \qquad (2.51) \\
R^{\bar{\lambda}}{}_{\bar{\mu}\nu\bar{\sigma}} &= \partial_{\nu}\Gamma^{\bar{\lambda}}{}_{\bar{\sigma}\bar{\mu}} = \partial_{\nu}(g^{\rho\bar{\lambda}}\partial_{\bar{\sigma}}g_{\rho\bar{\mu}}) \\
&= g^{\rho\bar{\lambda}}\partial_{\bar{\nu}}\partial_{\sigma}g_{\rho\bar{\mu}} + \partial_{\nu}g^{\rho\bar{\lambda}}\partial_{\bar{\sigma}}g_{\rho\bar{\mu}} \qquad (2.52)
\end{aligned}
$$

We note that $R^{\lambda}{}_{\mu\bar{\nu}\sigma}$ is the conjugate of $R^{\bar{\lambda}}{}_{\bar{\mu}\nu\bar{\sigma}}$, and $R^{\lambda}{}_{\mu\nu\bar{\sigma}}$ is the conjugate of $R^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}\sigma}$. Besides, we also have these obvious symmetries

$$R^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}\sigma} = -R^{\bar{\lambda}}{}_{\bar{\mu}\sigma\bar{\nu}}, \qquad R^{\lambda}{}_{\mu\bar{\nu}\sigma} = -R^{\lambda}{}_{\mu\sigma\bar{\nu}} \qquad (2.53)$$

so it is adequate for us to calculate only the term $R^{\bar{\lambda}}{}_{\bar{\mu}\nu\bar{\sigma}}$ as the components of Riemann tensor. We define

$$R_{\mu\bar{\nu}\lambda\bar{\sigma}} = g_{\mu\bar{\omega}}R^{\bar{\omega}}{}_{\bar{\nu}\lambda\bar{\sigma}} \qquad (2.54)$$

such that we have

$$\begin{aligned} R_{\mu\bar{\nu}\lambda\bar{\sigma}} &= g_{\mu\bar{\omega}}(g^{\rho\bar{\omega}}\partial_\lambda\partial_{\bar{\sigma}}g_{\rho\bar{\nu}} + \partial_\lambda g^{\rho\bar{\omega}}\partial_{\bar{\sigma}}g_{\rho\bar{\nu}}) \\ &= \partial_\lambda\partial_{\bar{\sigma}}g_{\mu\bar{\nu}} - g^{\rho\bar{\omega}}\partial_\lambda g_{\mu\bar{\omega}}\partial_{\bar{\sigma}}g_{\rho\bar{\nu}} \end{aligned} \tag{2.55}$$

It provides the details of Riemann curvature tensor completely.

Ricci Curvature Tensor

We continue the discussion to the next tensor which represents the curvature of complex manifolds, the *Ricci curvature tensor* $R_{\mu\bar{\nu}}$. It is defined as

$$R_{\mu\bar{\nu}} = R^{\lambda}{}_{\lambda\mu\bar{\nu}} \tag{2.56}$$

such that in terms of metric, it has the form

$$R_{\mu\bar{\nu}} = -\partial_{\bar{\nu}}g^{\lambda\bar{\rho}}\partial_\mu g_{\lambda\bar{\rho}} - g^{\lambda\bar{\rho}}\partial_{\bar{\nu}}\partial_\mu g_{\lambda\bar{\rho}} \tag{2.57}$$
$$= -\partial_{\bar{\nu}}(g^{\lambda\bar{\rho}}\partial_\mu g_{\lambda\bar{\rho}}) \tag{2.58}$$

or in terms of the determinant of metric, it's form is

$$R_{\mu\bar{\nu}} = -\partial_{\bar{\nu}}\partial_\mu \ln\det g_{\lambda\bar{\rho}} \tag{2.59}$$

By using the equation above we can find the Ricci tensor completely if we know the details of metric.

Ricci Scalar Curvature

If we contract the Ricci tensor $R_{\mu\bar{\nu}}$ once more, we will get the scalar, which is called *Ricci scalar curvature*, methematically it is written as

$$R = g^{\mu\bar{\nu}}R_{\mu\bar{\nu}} \tag{2.60}$$

Remark. If the complex manifold M is flat, then its Riemann curvature tensor vanishes ($R_{\mu\bar{\nu}\lambda\bar{\sigma}} = 0$). If the Riemann tensor is zero, then its Ricci tensor is also zero, but the converse is not always true. If this Ricci tensor vanishes ($R_{\mu\bar{\nu}} = 0$), then the complex manifold M is called *Ricci flat*. Furthermore, if its Ricci tensor is zero, then its Ricci scalar also vanishes, but the converse again is not true. So if the Riemann tensor is zero (or the manifold is flat), then its Ricci tensor and Ricci scalar are zero (and that is why we call the manifold *is* flat). But if the Ricci scalar is zero, nothing can be said about the Ricci and Riemann tensors.

Chapter 3

Solitons in the Kähler-Ricci Flow

3.1 Definition of Kähler-Ricci Solitons

Suppose we have a complete noncompact n-dimensional Kähler manifold M, with an initial Kähler metric $\tilde{g}_{i\bar{j}}(z)$, where $z \in M$. We can define the *Kähler-Ricci flow* as the partial differential equation which has the form

$$\frac{\partial}{\partial t} g_{i\bar{j}}(z,t) = -R_{i\bar{j}}(z,t) \tag{3.1}$$

defined on $M \times [0,\infty)$, where $(z,t) \in M \times [0,\infty)$, and $g_{i\bar{j}}(z,0) = \tilde{g}_{i\bar{j}}(z)$. If the solution of (3.1) flows along the one-parameter family of biholomorphisms

$$g_{i\bar{j}}(z,t) = \sigma(t)\varphi_t^* g_{i\bar{j}}(z,0) \tag{3.2}$$

where $\sigma(t) = 1 + \lambda t$, for a constant $\lambda \in \mathbb{R}$, is the scaling function, and the flow φ is generated along the negative direction of the holomorphic vector field V, then we will have the equation that characterizes the metric at $t = 0$, i.e.

$$\begin{aligned} \frac{\partial}{\partial t} g_{i\bar{j}}(z,t)\Big|_{t=0} &= \sigma'(0)\varphi_t^* g_{i\bar{j}}(z,0)\Big|_{t=0} - \sigma(0)\mathcal{L}_V g_{i\bar{j}}(z,0) \\ -R_{i\bar{j}}(z,0) &= \lambda g_{i\bar{j}}(z,0) - \mathcal{L}_V g_{i\bar{j}}(z,0) \end{aligned}$$

where $\mathcal{L}_V g_{i\bar{j}}(z,0)$ is the Lie derivative of metric $g_{i\bar{j}}(z,0)$ at $t = 0$. Then we have

$$R_{i\bar{j}}(z,0) + \lambda g_{i\bar{j}}(z,0) = \partial_i V_{\bar{j}} + \partial_{\bar{j}} V_i \tag{3.3}$$

The solution of (3.3) is called *Kähler-Ricci soliton*, and $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ cases correspond to *expanding*, *steady* and *shrinking Kähler-Ricci solitons*, respectively.

Conversely, if we have a metric that undergoes the biholomorphism condition (3.2), then by defining that

$$\sigma(t) \equiv 1 + \lambda t \tag{3.4}$$

and the vector field

$$W(t) \equiv \frac{1}{\sigma(t)} V \tag{3.5}$$

then the equation (3.2) becomes

$$\begin{aligned} \frac{\partial}{\partial t} g_{i\bar{j}}(z,t) &= \sigma'(t)\varphi_t^* g_{i\bar{j}}(z,0) - \sigma(t)\varphi_t^*(\mathcal{L}_W(t) g_{i\bar{j}}(z,0)) \\ &= \lambda\varphi_t^* g_{i\bar{j}}(z,0) - \varphi_t^*(\mathcal{L}_V g_{i\bar{j}}(z,0)) \\ &= \varphi_t^*(\lambda g_{i\bar{j}}(z,0) - \mathcal{L}_V g_{i\bar{j}}(z,0)) \\ &= \varphi_t^*(-R_{i\bar{j}}(z,0)) \\ &= -R_{i\bar{j}}(z,t) \end{aligned} \tag{3.6}$$

where we have used the equation (3.3). The calculation above is identic to equation (3.1), and hence establishes the equivalence.

If the vector field V is the gradient of a real-valued function f (which we can view as the 0-form on M), then equation (3.3) becomes

$$R_{i\bar{j}}(z,0) + \lambda g_{i\bar{j}}(z,0) = \partial_i \partial_{\bar{j}} f \tag{3.7}$$

and the holomorphicity of V is guaranteed by

$$\partial_i \partial_j f = 0 \tag{3.8}$$

The cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ in equation (3.7) correspond to *expanding*, *steady* and *shrinking gradient Kähler-Ricci solitons*, respectively.

3.2 Rotationally Invariant Solitons

Suppose we have a noncompact complex plane $\mathbb{C}^n$ equipped with the coordinate $z = \{z_1, \ldots, z_n\}$. The Kähler metric $g_{i\bar{j}}(z)$ can be described in terms of the Kähler potential $\Phi(z)$ as follows

$$g_{i\bar{j}}(z) = \partial_i \partial_{\bar{j}} \Phi(z) \tag{3.9}$$

We want to make a constraint on the potential $\Phi(z)$ such that it is rotationally symmetric

$$\Phi(z) \equiv u(|z|^2) \tag{3.10}$$

and by defining $s \equiv \ln |z|^2$, we can use the Kähler potential in the form $u(s)$, for $s \in (-\infty, \infty)$.

The function $u(s)$ must satisfy

$$u'(s) > 0, \qquad u''(s) > 0, \qquad \text{for } s \in (-\infty, \infty) \tag{3.11}$$

which is obtained by the positive-definiteness condition of the Kähler metric (which we will show below), and it also must satisfy the asymptotic condition at $s \to -\infty$,

$$u(s) = a_0 + a_1 e^s + a_2 e^{2s} + a_3 e^{3s} + \cdots, \qquad a_1 > 0 \tag{3.12}$$

From now on we will omit the argument of Kähler metric by remembering that we work in $t = 0$ for this section. By straightforward calculation we can find the explicit form of the Kähler metric $g_{i\bar{j}}$ in terms of Kähler potential

$$\begin{aligned} g_{i\bar{j}} &= \partial_i \partial_{\bar{j}} u(s) \\ &= \frac{\partial}{\partial z^i}\Big(u'(s) e^{-s} z_j\Big) \\ &= u'(s) e^{-s} \delta_{ij} + (u''(s) - u'(s)) e^{-2s} \bar{z}_i z_j \end{aligned} \tag{3.13}$$

It is important to note that the metric (3.13) above can be viewed as the sum of two different $n \times n$ matrices. Then our next task is to find out the inverse metric and the explicit form for its determinant.

Remember that if we are given

$$B = A + XY \tag{3.14}$$

where B and A are the $n \times n$ matrices, X is the $n \times 1$ matrix, and Y is the $1 \times n$ matrix, then the inverse of matrix B is

$$B^{-1} = A^{-1} - \frac{1}{1 + YA^{-1}X} A^{-1} XYA^{-1} \tag{3.15}$$

Therefore, by defining

$$p(s) \equiv u'(s)e^{-s}, \qquad q(s) \equiv (u''(s) - u'(s))e^{-2s} \tag{3.16}$$

and

$$X \equiv \sqrt{q(s)} \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix}, \qquad Y \equiv \sqrt{q(s)} \begin{pmatrix} z_1 & \cdots & z_n \end{pmatrix} \tag{3.17}$$

where X is the $n \times 1$ matrix and Y is the $1 \times n$ matrix, the equation (3.13) can be written as

$$g_{i\bar{j}} = p(s)\delta_{ij} + XY \tag{3.18}$$

such that we can find the explicit form for its inverse

$$\begin{aligned} g^{i\bar{j}} &= \frac{1}{p(s)}\delta^{ij} - \frac{1}{1 + \frac{q(s)}{p(s)} \sum_i z_i^2} \frac{1}{(p(s))^2} z^i \bar{z}^j \\ &= \frac{1}{u'(s)} e^s \delta^{ij} - \frac{(u''(s) - u'(s))e^{-2s}}{(u'(s))^2 e^{-2s} + u'(s)(u''(s) - u'(s))e^{-2s}} z^i \bar{z}^j \\ &= (u'(s))^{-1} e^s \delta^{ij} + ((u''(s))^{-1} - (u'(s))^{-1}) z^i \bar{z}^j \end{aligned} \tag{3.19}$$

and for the determinant of metric, which is also straightforwadly easy to compute

$$\begin{aligned} \det g_{i\bar{j}} &= (p(s))^n + (p(s))^{n-1} q(s)|z|^2 \\ &= e^{-ns}(u'(s))^n + e^{-(n-1)s}(u'(s))^{n-1}(u''(s) - u'(s))e^{-s} \\ &= e^{-ns}(u'(s))^{n-1} u''(s) \end{aligned} \tag{3.20}$$

where we can see that the positive-definiteness of the Kähler metric implies the constraint (3.11).

Define

$$\begin{aligned} w(s) &\equiv -\ln \det g_{i\bar{j}} \\ &= ns - (n-1)\ln u'(s) - \ln u''(s) \end{aligned} \tag{3.21}$$

then the Ricci curvature tensor can be written as

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} w(s) \tag{3.22}$$

such that the equation (3.7) becomes

$$R_{i\bar{j}} + \lambda g_{i\bar{j}} = \partial_i \partial_{\bar{j}} (w(s) + \lambda u(s)) \tag{3.23}$$

where we have identified $f(s) = w(s) + \lambda u(s)$.

The holomorphic vector field V has the form

$$\begin{aligned} V^i &= g^{i\bar{j}} \partial_{\bar{j}} (w + \lambda u) \\ &= \Big((u')^{-1} e^s \delta^{ij} + ((u'')^{-1} - (u')^{-1}) z^i \bar{z}^j \Big) (w' + \lambda u') e^{-s} z_j \\ &= \frac{w' + \lambda u'}{u''} z^i \end{aligned} \tag{3.24}$$

Since the vector V is holomorphic, there must exist $\mu \in \mathbb{R}$ such that

$$w' + \lambda u' = -\mu u'' \tag{3.25}$$

and by subtituting equation (3.21) to (3.25), we will get

$$\begin{aligned} n - (n-1)\frac{u''}{u'} - \frac{u'''}{u''} + \lambda u' &= -\mu u'' \\ \frac{u'''}{u''} + \Big(\frac{n-1}{u'} - \mu \Big) u'' &= n + \lambda u' \end{aligned} \tag{3.26}$$

Now define

$$\phi(s) \equiv u'(s) \tag{3.27}$$

then equation (3.26) becomes

$$\frac{\phi''}{\phi'} + \Big(\frac{n-1}{\phi} - \mu \Big) \phi' = n + \lambda \phi \tag{3.28}$$

or, by modifying the first term in previous equation,

$$\frac{d\phi'}{d\phi} + \Big(\frac{n-1}{\phi} - \mu \Big) \phi' = n + \lambda \phi \tag{3.29}$$

By defining

$$A(\phi) \equiv \frac{n-1}{\phi} - \mu, \qquad B(\phi) \equiv n + \lambda \phi \tag{3.30}$$

equation (3.28) can be calculated easily, to get the result

$$\begin{aligned} \frac{d\phi'}{d\phi} + A(\phi)\phi' &= B(\phi) \\ e^{\int A(\phi)\, d\phi} \frac{d\phi'}{d\phi} + e^{\int A(\phi)\, d\phi} A(\phi)\phi' &= e^{\int A(\phi)\, d\phi} B(\phi) \\ \frac{d}{d\phi} \Big(\phi' e^{\int A(\phi)\, d\phi} \Big) &= e^{\int A(\phi)\, d\phi} B(\phi) \\ \phi' e^{\int A(\phi)\, d\phi} &= \int e^{\int A(\phi)\, d\phi} B(\phi)\, d\phi \end{aligned} \tag{3.31}$$

such that we will get

$$\phi' = e^{-\int A(\phi)\,d\phi} \int e^{\int A(\phi)\,d\phi} B(\phi)\,d\phi \tag{3.32}$$

Since

$$\int A(\phi)\,d\phi = \int \Big(\frac{n-1}{\phi} - \mu\Big)\,d\phi = (n-1)\ln\phi - \mu\phi \tag{3.33}$$

and consequently $e^{\int A(\phi)\,d\phi} = \phi^{n-1}e^{-\mu\phi}$, we have from equation (3.32),

$$\begin{aligned}\phi' &= \phi^{1-n}e^{\mu\phi}\int \phi^{n-1}e^{-\mu\phi}(n+\lambda\phi)\,d\phi \\ &= \phi^{1-n}e^{\mu\phi}\Big(n\int \phi^{n-1}e^{-\mu\phi}\,d\phi + \lambda\int \phi^{n}e^{-\mu\phi}\,d\phi\Big)\end{aligned} \tag{3.34}$$

Define

$$I_n \equiv \int \phi^n e^{-\mu\phi}\,d\phi \tag{3.35}$$

then by partial integration we have the recursive relation for I_n,

$$\begin{aligned}I_n &= -\frac{1}{\mu}\phi^n e^{-\mu\phi} + \int \frac{n}{\mu}\phi^{n-1}e^{-\mu\phi}\,d\phi \\ &= -\frac{1}{\mu}\phi^n e^{-\mu\phi} + \frac{n}{\mu}I_{n-1}\end{aligned} \tag{3.36}$$

and by continuing this relation, we have

$$\begin{aligned}I_n &= -\frac{1}{\mu}\phi^n e^{-\mu\phi} + \frac{n}{\mu}I_{n-1} \\ &= -\frac{1}{\mu}\phi^n e^{-\mu\phi} - \frac{n}{\mu^2}\phi^{n-1}e^{-\mu\phi} + \frac{n(n-1)}{\mu^2}I_{n-2} \\ &= -\frac{1}{\mu}\phi^n e^{-\mu\phi} - \frac{n}{\mu^2}\phi^{n-1}e^{-\mu\phi} \\ &\quad -\frac{n(n-1)}{\mu^3}\phi^{n-2}e^{-\mu\phi} - \frac{n(n-1)(n-2)}{\mu^3}I_{n-3} \\ &= -\sum_{k=0}^{n}\frac{1}{\mu^{k+1}}\frac{n!}{(n-k)!}\phi^{n-k}e^{-\mu\phi} \\ &= -\sum_{k=0}^{n}\frac{1}{\mu^{n-k+1}}\frac{n!}{k!}\phi^{k}e^{-\mu\phi}\end{aligned} \tag{3.37}$$

where we get the explicit form for I_n.

Equation (3.34) in terms of I_n is

$$\phi' = \phi^{1-n}e^{\mu\phi}(nI_{n-1} + \lambda I_n + c) \tag{3.38}$$

for a constant $c \in \mathbb{R}$. Hence, by inserting equation (3.37) to (3.38), it becomes

$$\begin{aligned}
\phi' &= \phi^{1-n} e^{\mu\phi} \Bigg(-n \sum_{k=0}^{n-1} \frac{1}{\mu^{n-k}} \frac{(n-1)!}{k!} \phi^k e^{-\mu\phi} \\
&\qquad -\lambda \sum_{k=0}^{n} \frac{1}{\mu^{n-k+1}} \frac{n!}{k!} \phi^k e^{-\mu\phi} + c \Bigg) \\
&= \phi^{1-n} e^{\mu\phi} \Bigg(-\frac{\lambda}{\mu} \phi^n e^{-\mu\phi} - \frac{\lambda+\mu}{\mu^{n+1}} \sum_{k=0}^{n-1} \frac{n!}{k!} \mu^k \phi^k e^{-\mu\phi} + c \Bigg) \\
&= -\frac{\lambda}{\mu}\phi - \frac{\lambda+\mu}{\mu^{n+1}} \sum_{k=0}^{n-1} \frac{n!}{k!} \mu^k \phi^{k+1-n} + \frac{c e^{\mu\phi}}{\phi^{n+1}} \qquad (3.39)
\end{aligned}$$

The equation (3.39) above is the ODE which is satisfied by the rotationally invariant gradient Kähler-Ricci soliton on the complex plane $\mathbb{C}^n$.

Chapter 4

Calculation and Result

4.1 The Curvatures of Rotationally Invariant Kähler Metric

We can study the richness of properties of our soliton by analyzing the structure of its curvatures. Since the soliton in section 3.2 is built to be rotationally invariant, we should calculate the curvatures for this model, by first computing the curvatures for any rotationally invariant Kähler metric and restricting the results for the case of rotationally invariant soliton using equation (3.28).

For the next three lemmas we assume that we are working on the Kähler manifold M attached by the coordinate system $\{z^i\}$ and equipped with Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u(s)$, where $u(s)$ is the Kähler potential and $s \equiv \ln |z|^2$, and with $u' \equiv \phi$. The curvatures of this metric are described by Riemann tensor $R_{i\bar{j}k\bar{l}}$, Ricci tensor $R_{i\bar{j}}$, and the Ricci scalar R.

Lemma 1. *The Riemann tensor $R_{i\bar{j}k\bar{l}}$ for the rotationally invariant Kähler* *metric $g_{i\bar{j}}$ is given by*

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= (\phi''' - 6\phi'' + 11\phi' - 6\phi)e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +(\phi'' - 3\phi' + 2\phi)e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{kl}\bar{z}_i z_j + \delta_{jk}\bar{z}_i z_l) \\
&\quad +(\phi' - \phi)e^{-2s}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \\
&\quad -\frac{1}{\phi}(\phi' - \phi)^2 e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{jk}\bar{z}_i z_l + \delta_{kl}\bar{z}_i z_j) \\
&\quad -4\Big(\frac{1}{\phi'} - \frac{1}{\phi}\Big)(\phi' - \phi)^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad -\Big(1 - 2\frac{\phi}{\phi'} + \frac{\phi''}{\phi'}\Big)(\phi'' - 3\phi' + 2\phi)e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \qquad (4.1)
\end{aligned}$$

Proof. The Riemann curvature tensor of the Kähler metric $g_{i\bar{j}}$ has the form stated in equation (2.55) above as

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} - g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l} \qquad (4.2)$$

The first term on the RHS of equation (4.2) above can be calculated easily as follows

$$\begin{aligned}
\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} &= \frac{\partial}{\partial z^k}\frac{\partial}{\partial \bar{z}^l}\Big(u' e^{-s}\delta_{i\bar{j}} + (u'' - u')e^{-2s}\bar{z}_i z_j\Big) \\
&= \frac{\partial}{\partial z^k}\Big((u'' - u')e^{-2s}(\delta_{ij}z_l + \delta_{il}z_j) + (u''' - 3u'' + 2u')e^{-3s}\bar{z}_i z_j z_l\Big) \\
&= \Big((u''' - u'')e^{-2s} - 2(u'' - u')e^{-2s}\Big)e^{-s}\bar{z}_k(\delta_{ij}z_l + \delta_{il}z_j) \\
&\quad +(u'' - u')e^{-2s}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) + \Big((u'''' - 3u''' + 2u'')e^{-3s} \\
&\quad -3(u''' - 3u'' + 2u')e^{-3s}\Big)e^{-s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +(u''' - 3u'' + 2u')e^{-3s}\bar{z}_i(\delta_{kl}z_j + \delta_{jk}z_l) \\
&= (u''' - 3u'' + 2u')e^{-3s}\bar{z}_k(\delta_{ij}z_l + \delta_{il}z_j) \\
&\quad +(u'' - u')e^{-2s}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \\
&\quad +(u'''' - 6u''' + 11u'' - 6u')e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +(u''' - 3u'' + 2u')e^{-3s}\bar{z}_i(\delta_{kl}z_j + \delta_{jk}z_l) \\
&= (u'''' - 6u''' + 11u'' - 6u')e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +(u''' - 3u'' + 2u')e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{kl}\bar{z}_i z_j + \delta_{jk}\bar{z}_i z_l) \\
&\quad +(u'' - u')e^{-2s}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \qquad (4.3)
\end{aligned}$$

And the second term on the RHS of equation (4.2) can be calculated straightforwardly as

$$\begin{aligned}
g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l} &= g^{p\bar{q}}\frac{\partial}{\partial z^k}(u'e^{-s}\delta_{iq}+(u''-u')e^{-2s}\bar{z}_i z_q) \\
&\quad \frac{\partial}{\partial \bar{z}^l}(u'e^{-s}\delta_{pj}+(u''-u')e^{-2s}\bar{z}_p z_j) \\
&= g^{p\bar{q}}\Big((u''-u')e^{-2s}\bar{z}_k\delta_{iq} \\
&\quad +(u'''-3u''+2u')e^{-3s}\bar{z}_i\bar{z}_k z_q+(u''-u')e^{-2s}\bar{z}_i\delta_{kq}\Big) \\
&\quad \Big((u''-u')e^{-2s}z_l\delta_{pj}+(u'''-3u''+2u')e^{-3s}z_j z_l\bar{z}_p \\
&\quad +(u''-u')e^{-2s}z_j\delta_{pl}\Big) \\
&= \Big(\frac{1}{u'}e^s\delta^{pq}+\Big(\frac{1}{u''}-\frac{1}{u'}\Big)z^p\bar{z}^q\Big)\Big((u''-u')e^{-2s}(\delta_{iq}\bar{z}_k+\delta_{kq}\bar{z}_i) \\
&\quad +(u'''-3u''+2u')e^{-3s}\bar{z}_i\bar{z}_k z_q\Big)\Big((u''-u')e^{-2s}(\delta_{pj}z_l+\delta_{pl}z_j) \\
&\quad +(u'''-3u''+2u')e^{-3s}z_j z_l\bar{z}_p\Big) \\
&= \Big(\frac{1}{u'}e^s\delta^{pq}+\Big(\frac{1}{u''}-\frac{1}{u'}\Big)z^p\bar{z}^q\Big) \\
&\quad \Big((u''-u')^2e^{-4s}(\delta_{iq}\delta_{pj}\bar{z}_k z_l+\delta_{iq}\delta_{pl}\bar{z}_k z_j+\delta_{kq}\delta_{pj}\bar{z}_i z_l \\
&\quad +\delta_{kq}\delta_{pl}\bar{z}_i z_j) \\
&\quad +(u''-u')(u'''-3u''+2u')e^{-5s}(\delta_{iq}\bar{z}_k\bar{z}_p z_j z_l+\delta_{kq}\bar{z}_i\bar{z}_p z_j z_l \\
&\quad +\delta_{pj}\bar{z}_i\bar{z}_k z_l z_q+\delta_{pl}\bar{z}_i\bar{z}_k z_j z_q) \\
&\quad +(u'''-3u''+2u')^2e^{-6s}\bar{z}_i\bar{z}_k\bar{z}_p z_j z_l z_q\Big) \\
&= \frac{1}{u'}(u''-u')^2e^{-3s}(\delta_{ij}\bar{z}_k z_l+\delta_{il}\bar{z}_k z_j+\delta_{jk}\bar{z}_i z_l+\delta_{kl}\bar{z}_i z_j) \\
&\quad +\frac{4}{u'}(u''-u')(u'''-3u''+2u')e^{-4s}\bar{z}_i\bar{z}_k z_j z_l \\
&\quad +\frac{1}{u'}(u'''-3u''+2u')^2e^{-4s}\bar{z}_i z_j\bar{z}_k z_l \\
&\quad +4\Big(\frac{1}{u''}-\frac{1}{u'}\Big)(u''-u')^2e^{-4s}\bar{z}_i z_j\bar{z}_k z_l \\
&\quad +4\Big(\frac{1}{u''}-\frac{1}{u'}\Big)(u''-u')(u'''-3u''+2u')e^{-4s}\bar{z}_i z_j\bar{z}_k z_l \\
&\quad +\Big(\frac{1}{u''}-\frac{1}{u'}\Big)(u'''-3u''+2u')^2e^{-4s}\bar{z}_i z_j\bar{z}_k z_l
\end{aligned}$$

where by simplifying the terms, we will have

$$
\begin{aligned}
g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l} &= \frac{1}{u'}(u''-u')^2 e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{jk}\bar{z}_i z_l + \delta_{kl}\bar{z}_i z_j) \\
&\quad +4\Big(\frac{1}{u''}-\frac{1}{u'}\Big)(u''-u')^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +\left(4\Big(\frac{u''}{u'}-1\Big)+4\Big(1-\frac{u''}{u'}-\frac{u'}{u''}+1\Big)\right)(u'''-3u''+2u') \\
&\qquad e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +\left(\frac{1}{u'}+\Big(\frac{1}{u''}-\frac{1}{u'}\Big)\right)(u'''-3u''+2u')^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&= \frac{1}{u'}(u''-u')^2 e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{jk}\bar{z}_i z_l + \delta_{kl}\bar{z}_i z_j) \\
&\quad +4\Big(\frac{1}{u''}-\frac{1}{u'}\Big)(u''-u')^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +4\Big(1-\frac{u'}{u''}\Big)(u'''-3u''+2u')e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +\frac{1}{u''}(u'''-3u''+2u')^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \qquad (4.4)
\end{aligned}
$$

By using the convention (3.27) we can write the two terms in the RHS of equation (4.2) above as

$$
\begin{aligned}
\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} &= (\phi'''-6\phi''+11\phi'-6\phi)e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +(\phi''-3\phi'+2\phi)e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{kl}\bar{z}_i z_j + \delta_{jk}\bar{z}_i z_l) \\
&\quad +(\phi'-\phi)e^{-2s}(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}) \qquad (4.5) \\
g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l} &= \frac{1}{\phi}(\phi'-\phi)^2 e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{jk}\bar{z}_i z_l + \delta_{kl}\bar{z}_i z_j) \\
&\quad +4\Big(\frac{1}{\phi'}-\frac{1}{\phi}\Big)(\phi'-\phi)^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +4\Big(1-\frac{\phi}{\phi'}\Big)(\phi''-3\phi'+2\phi)e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +\frac{1}{\phi'}(\phi''-3\phi'+2\phi)^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&= \frac{1}{\phi}(\phi'-\phi)^2 e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{jk}\bar{z}_i z_l + \delta_{kl}\bar{z}_i z_j) \\
&\quad +4\Big(\frac{1}{\phi'}-\frac{1}{\phi}\Big)(\phi'-\phi)^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +\Big(1-2\frac{\phi}{\phi'}+\frac{\phi''}{\phi'}\Big)(\phi''-3\phi'+2\phi)e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \qquad (4.6)
\end{aligned}
$$

Therefore, we have the explicit form of the Riemann curvature tensor as

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= (\phi''' - 6\phi'' + 11\phi' - 6\phi)e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad +(\phi'' - 3\phi' + 2\phi)e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{kl}\bar{z}_i z_j + \delta_{jk}\bar{z}_i z_l) \\
&\quad +(\phi' - \phi)e^{-2s}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \\
&\quad -\frac{1}{\phi}(\phi' - \phi)^2 e^{-3s}(\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{jk}\bar{z}_i z_l + \delta_{kl}\bar{z}_i z_j) \\
&\quad -4\Big(\frac{1}{\phi'} - \frac{1}{\phi}\Big)(\phi' - \phi)^2 e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \\
&\quad -\Big(1 - 2\frac{\phi}{\phi'} + \frac{\phi''}{\phi'}\Big)(\phi'' - 3\phi' + 2\phi)e^{-4s}\bar{z}_i z_j \bar{z}_k z_l \qquad (4.7)
\end{aligned}$$

which completes the proof of this lemma. □

Since our metric is rotationally symmetric, then it is adequate to find the description for Riemann curvature tensor at point $P = \{z_1, 0, \ldots, 0\}$. At this point,

$$|z|^2 = z_1\bar{z}_1 = e^s, \qquad s = \ln|z|^2 \qquad (4.8)$$

$$\bar{z}_i z_j \bar{z}_k z_l = \delta_{ijkl1}e^{2s} \qquad (4.9)$$

$$\delta_{ij}\bar{z}_k z_l + \delta_{il}\bar{z}_k z_j + \delta_{kl}\bar{z}_i z_j + \delta_{jk}\bar{z}_i z_l = (\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{kj1} + \delta_{kl}\delta_{ij1} + \delta_{jk}\delta_{il1})e^s \qquad (4.10)$$

where δ_{ijkl1} and δ_{ij1} are zero unless all indices are 1. Using these three equations we can write the equation (4.1) above as

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= (\phi''' - 6\phi'' + 11\phi' - 6\phi)e^{-2s}\delta_{ijkl1} \\
&\quad +(\phi'' - 3\phi' + 2\phi)e^{-2s}(\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{kj1} + \delta_{kl}\delta_{ij1} + \delta_{jk}\delta_{il1}) \\
&\quad +(\phi' - \phi)e^{-2s}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \\
&\quad -\frac{1}{\phi}(\phi' - \phi)^2 e^{-2s}(\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{kj1} + \delta_{kl}\delta_{ij1} + \delta_{jk}\delta_{il1}) \\
&\quad -4\Big(\frac{1}{\phi'} - \frac{1}{\phi}\Big)(\phi' - \phi)^2 e^{-2s}\delta_{ijkl1} \\
&\quad -\Big(1 - 2\frac{\phi}{\phi'} + \frac{\phi''}{\phi'}\Big)(\phi'' - 3\phi' + 2\phi)e^{-2s}\delta_{ijkl1} \\
&= (\phi' - \phi)e^{-2s}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \\
&\quad +(\phi'' - 3\phi' + 2\phi)e^{-2s}(\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{kj1} + \delta_{kl}\delta_{ij1} + \delta_{jk}\delta_{il1}) \\
&\quad +(\phi''' - 6\phi'' + 11\phi' - 6\phi)e^{-2s}\delta_{ijkl1} \\
&\quad -\frac{1}{\phi}(\phi' - \phi)^2 e^{-2s}(\delta_{ij\hat{1}}\delta_{kl1} + \delta_{il\hat{1}}\delta_{kj1} + \delta_{kl\hat{1}}\delta_{ij1} + \delta_{jk\hat{1}}\delta_{il1}) \\
&\quad -\frac{4}{\phi'}(\phi' - \phi)^2 e^{-2s}\delta_{ijkl1} \\
&\quad -\frac{1}{\phi'}(\phi'' + \phi' - 2\phi)(\phi'' - 3\phi' + 2\phi)e^{-2s}\delta_{ijkl1} \qquad (4.11)
\end{aligned}$$

where $\delta_{ij\hat{1}}$ in equation (4.11) above means zero unless $i = j$ and neither i or j is 1.

The last term in equation (4.11) can be computed to get the relation

$$\frac{1}{\phi'}(\phi'' + \phi' - 2\phi)(\phi'' - 3\phi' + 2\phi)e^{-2s}\delta_{ijkl1} = \frac{1}{\phi'}(\phi'' - \phi')^2 e^{-2s}\delta_{ijkl1} \qquad (4.12)$$

such that we will have the simpler description for Riemann curvature tensor of rotationally invariant Kähler metric on $\mathbb{C}^n$

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= \Big((\phi' - \phi)(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \\
&\quad +(\phi'' - 3\phi' + 2\phi)(\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{kj1} + \delta_{kl}\delta_{ij1} + \delta_{jk}\delta_{il1}) \\
&\quad +(\phi''' - 6\phi'' + 11\phi' - 6\phi)\delta_{ijkl1} \\
&\quad -\frac{1}{\phi}(\phi' - \phi)^2(\delta_{ij\hat{1}}\delta_{kl1} + \delta_{il\hat{1}}\delta_{kj1} + \delta_{kl\hat{1}}\delta_{ij1} + \delta_{jk\hat{1}}\delta_{il1}) \\
&\quad -\frac{1}{\phi'}(\phi'' - \phi')^2\delta_{ijkl1}\Big)e^{-2s} \qquad (4.13)
\end{aligned}$$

The author of [2] has proved that by using the equation above for Riemann tensor, the sectional curvature of metric in manifold M is negative.

Next we compute the very important curvature tensor, the Ricci tensor $R_{i\bar{j}}$.

Lemma 2. *The Ricci tensor $R_{i\bar{j}}$ for the rotationally invariant Kähler metric $g_{i\bar{j}}$ is given by*

$$\begin{aligned} R_{i\bar{j}} &= \left(-\frac{\phi''}{\phi'}-(n-1)\frac{\phi'}{\phi}+n\right)e^{-s}\delta_{ij} \\ &\quad +\left(-\frac{\phi'''}{\phi'}+\frac{\phi''^2}{\phi'^2}+\frac{\phi''}{\phi'}-(n-1)\frac{\phi''}{\phi}+(n-1)\frac{\phi'^2}{\phi^2}\right. \\ &\quad \left.+(n-1)\frac{\phi'}{\phi}-n\right)e^{-2s}\bar{z}_i z_j \end{aligned} \tag{4.14}$$

Proof. Now we should calculate each terms in the RHS of equation (2.57).

$$\begin{aligned} \partial_i g_{k\bar{p}} &= \partial_i(u'e^{-s}\delta_{kp}+(u''-u')e^{-2s}\bar{z}_k z_p) \\ &= (u''e^{-s}-u'e^{-s})e^{-s}\bar{z}_i\delta_{kp}+\Big((u'''-u'')e^{-2s} \\ &\quad -2(u''-u')e^{-2s}\Big)e^{-s}\bar{z}_i\bar{z}_k z_p+(u''-u')e^{-2s}\bar{z}_k\delta_{ip} \\ &= (u''-u')e^{-2s}(\bar{z}_i\delta_{kp}+\bar{z}_k\delta_{ip}) \\ &\quad +(u'''-3u''+2u')e^{-3s}\bar{z}_i\bar{z}_k z_p \end{aligned} \tag{4.15}$$

such that from the result above we can calculate

$$\begin{aligned} \partial_{\bar{j}}\partial_i g_{k\bar{p}} &= \partial_{\bar{j}}\Big((u''-u')e^{-2s}(\bar{z}_i\delta_{kp}+\bar{z}_k\delta_{ip})+(u'''-3u''+2u')e^{-3s}\bar{z}_i\bar{z}_k z_p\Big) \\ &= \Big((u'''-u'')e^{-2s}-2(u''-u')e^{-2s}\Big)e^{-s}z_j(\bar{z}_i\delta_{kp}+\bar{z}_k\delta_{ip}) \\ &\quad +(u''-u')e^{-2s}(\delta_{ij}\delta_{kp}+\delta_{jk}\delta_{ip})+\Big((u''''-3u'''+2u'')e^{-3s} \\ &\quad -3(u'''-3u''+2u')e^{-3s}\Big)e^{-s}\bar{z}_i z_j\bar{z}_k z_p \\ &\quad +(u'''-3u''+2u')e^{-3s}(\delta_{ij}\bar{z}_k z_p+\delta_{jk}\bar{z}_i z_p) \\ &= (u''-u')e^{-2s}(\delta_{ij}\delta_{kp}+\delta_{jk}\delta_{ip}) \\ &\quad +(u'''-3u''+2u')e^{-3s}(\delta_{kp}\bar{z}_i z_j+\delta_{ip}\bar{z}_k z_j+\delta_{ij}\bar{z}_k z_p+\delta_{jk}\bar{z}_i z_p) \\ &\quad +(u''''-6u'''+11u''-6u')e^{-4s}\bar{z}_i z_j\bar{z}_k z_p \end{aligned} \tag{4.16}$$

then we calculate this term

$$
\begin{aligned}
\partial_{\bar{j}} g^{k\bar{p}} &= \partial_{\bar{j}}\Big(\frac{1}{u'}e^{s}\delta^{kp} + \Big(\frac{1}{u''} - \frac{1}{u'}\Big)z^{k}\bar{z}^{p}\Big) \\
&= \Big(-\frac{u''}{u'^{2}}e^{s} + \frac{1}{u'}e^{s}\Big)e^{-s}z_{j}\delta^{kp} + \Big(-\frac{u'''}{u''^{2}} + \frac{u''}{u'^{2}}\Big)e^{-s}z_{j}z^{k}\bar{z}^{p} \\
&\quad +\Big(\frac{1}{u''} - \frac{1}{u'}\Big)z^{k}\delta^{p}{}_{j} \\
&= \frac{u''}{u'}\Big(\frac{1}{u''} - \frac{1}{u'}\Big)z_{j}\delta^{kp} + \Big(-\frac{u'''}{u''^{2}} + \frac{u''}{u'^{2}}\Big)e^{-s}z_{j}z^{k}\bar{z}^{p} \\
&\quad +\Big(\frac{1}{u''} - \frac{1}{u'}\Big)z^{k}\delta^{p}{}_{j} \\
&= \Big(\frac{1}{u''} - \frac{1}{u'}\Big)\Big(\frac{u''}{u'}z_{j}\delta^{kp} + z^{k}\delta^{p}{}_{j}\Big) \\
&\quad +\Big(-\frac{u'''}{u''^{2}} + \frac{u''}{u'^{2}}\Big)e^{-s}z_{j}z^{k}\bar{z}^{p} \qquad (4.17)
\end{aligned}
$$

such that we can calculate the second term in the RHS of equation (2.57)

$$
\begin{aligned}
g^{k\bar{p}}\partial_{\bar{j}}\partial_{i}g_{k\bar{p}} &= \Big(\frac{1}{u'}e^{s}\delta^{kp} + \Big(\frac{1}{u''} - \frac{1}{u'}\Big)z^{k}\bar{z}^{p}\Big)\Big((u''-u')e^{-2s}(\delta_{ij}\delta_{kp} + \delta_{jk}\delta_{ip}) \\
&\quad +(u'''-3u''+2u')e^{-3s}(\bar{z}_{i}z_{j}\delta_{kp} + \bar{z}_{k}z_{j}\delta_{ip} + \bar{z}_{k}z_{p}\delta_{ij} + \bar{z}_{i}z_{p}\delta_{jk}) \\
&\quad +(u''''-6u'''+11u''-6u')e^{-4s}\bar{z}_{i}z_{j}\bar{z}_{k}z_{p}\Big) \\
&= (n+1)\frac{1}{u'}(u''-u')e^{-s}\delta_{ij} + \frac{1}{u'}(u'''-3u''+2u')e^{-s}\delta_{ij} \\
&\quad +(n+2)\frac{1}{u'}(u'''-3u''+2u')e^{-2s}\bar{z}_{i}z_{j} + \frac{1}{u'}(u''''-6u''' \\
&\quad +11u''-6u')e^{-2s}\bar{z}_{i}z_{j} + \Big(\frac{1}{u''} - \frac{1}{u'}\Big)(u''-u')e^{-s}\delta_{ij} \\
&\quad +\Big(\frac{1}{u''} - \frac{1}{u'}\Big)(u''-u')e^{-2s}\bar{z}_{i}z_{j} + \Big(\frac{1}{u''} - \frac{1}{u'}\Big)(u'''-3u'' \\
&\quad +2u')e^{-s}\delta_{ij} + 3\Big(\frac{1}{u''} - \frac{1}{u'}\Big)(u'''-3u''+2u')e^{-2s}\bar{z}_{i}z_{j} \\
&\quad +\Big(\frac{1}{u''} - \frac{1}{u'}\Big)(u''''-6u'''+11u''-6u')e^{-2s}\bar{z}_{i}z_{j} \qquad (4.18)
\end{aligned}
$$

and we can simplify the result above to get

$$
\begin{aligned}
g^{k\bar{p}}\partial_{\bar{j}}\partial_{i}g_{k\bar{p}} &= \Big(\frac{u'''}{u''} + n\frac{u''}{u'} + \frac{u'}{u''} - (n+2)\Big)e^{-s}\delta_{ij} \\
&\quad +\Big(\frac{u''''}{u''} - 3\frac{u'''}{u''} + (n-1)\frac{u'''}{u'} + (-3n+2)\frac{u''}{u'} - \frac{u'}{u''} \\
&\quad +2(n+1)\Big)e^{-2s}\bar{z}_{i}z_{j} \qquad (4.19)
\end{aligned}
$$

The first term is

$$
\begin{aligned}
\partial_{\bar{j}} g^{k\bar{p}} \partial_i g_{k\bar{p}} &= \left(\left(\frac{1}{u''} - \frac{1}{u'} \right) \left(\frac{u''}{u'} z_j \delta^{kp} + z^k \delta^p{}_j \right) + \left(- \frac{u'''}{u''^2} + \frac{u''}{u'^2} \right) e^{-s} z_j z^k \bar{z}^p \right) \\
& \quad \left((u'' - u') e^{-2s} (\bar{z}_i \delta_{kp} + \bar{z}_k \delta_{ip}) + (u''' - 3u'' + 2u') e^{-3s} \bar{z}_i \bar{z}_k z_p \right) \\
&= \left(\frac{1}{u''} - \frac{1}{u'} \right) (u'' - u') e^{-2s} \left(n \frac{u''}{u'} \bar{z}_i z_j + \frac{u''}{u'} \bar{z}_i z_j + \bar{z}_i z_j + e^s \delta_{ij} \right) \\
& \quad + \left(\frac{1}{u''} - \frac{1}{u'} \right) (u''' - 3u'' + 2u') e^{-3s} \left(\frac{u''}{u'} \bar{z}_i z_j e^s + \bar{z}_i z_j e^s \right) \\
& \quad + 2(u'' - u') \left(- \frac{u'''}{u''^2} + \frac{u''}{u'^2} \right) e^{-2s} \bar{z}_i z_j \\
& \quad + (u''' - 3u'' + 2u') \left(- \frac{u'''}{u''^2} + \frac{u''}{u'^2} \right) e^{-2s} \bar{z}_i z_j \\
&= \left(\frac{1}{u''} - \frac{1}{u'} \right) (u'' - u') e^{-s} \delta_{ij} + \left(- \frac{u'''^2}{u''^2} + 2 \frac{u'''}{u''} + (-n+1) \frac{u''^2}{u'^2} \right. \\
& \quad \left. + (2n-1) \frac{u''}{u'} + \frac{u'}{u''} - (n+2) \right) e^{-2s} \bar{z}_i z_j \qquad (4.20)
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
R_{i\bar{j}} &= -\partial_{\bar{j}} g^{k\bar{p}} \partial_i g_{k\bar{p}} - g^{k\bar{p}} \partial_{\bar{j}} \partial_i g_{k\bar{p}} \\
&= - \left(\frac{1}{u''} - \frac{1}{u'} \right) (u'' - u') e^{-s} \delta_{ij} - \left(- \frac{u'''^2}{u''^2} + 2 \frac{u'''}{u''} + (-n+1) \frac{u''^2}{u'^2} \right. \\
& \quad \left. + (2n-1) \frac{u''}{u'} + \frac{u'}{u''} - (n+2) \right) e^{-2s} \bar{z}_i z_j - \left(\frac{u'''}{u''} + n \frac{u''}{u'} + \frac{u'}{u''} \right. \\
& \quad \left. - (n+2) \right) e^{-s} \delta_{ij} - \left(\frac{u''''}{u''} - 3 \frac{u'''}{u''} + (n-1) \frac{u'''}{u'} + (-3n+2) \frac{u''}{u'} \right. \\
& \quad \left. - \frac{u'}{u''} + 2(n+1) \right) e^{-2s} \bar{z}_i z_j \\
&= \left(- \frac{\phi''}{\phi'} - (n-1) \frac{\phi'}{\phi} + n \right) e^{-s} \delta_{ij} \\
& \quad + \left(- \frac{\phi'''}{\phi'} + \frac{\phi''^2}{\phi'^2} + \frac{\phi''}{\phi'} - (n-1) \frac{\phi''}{\phi} + (n-1) \frac{\phi'^2}{\phi^2} \right. \\
& \quad \left. + (n-1) \frac{\phi'}{\phi} - n \right) e^{-2s} \bar{z}_i z_j
\end{aligned}
$$

which is the complete description for the Ricci tensor. □

The formula for Ricci tensor above is valid for all rotationally symmetric Kähler metric. Therefore, we can examine the Ricci tensor only at a point

$P = \{z_1, 0, \ldots, 0\}$. Here we have $\bar{z}_i z_j = e^s \delta_{ij1}$, where the δ_{ij1} is zero unless $i = j = 1$. Define two functions $\alpha(s)$ and $\beta(s)$ as

$$\alpha(s) \equiv -\frac{\phi''}{\phi'} - (n-1)\frac{\phi'}{\phi} + n \tag{4.21}$$

$$\begin{aligned}\beta(s) \equiv & -\frac{\phi'''}{\phi'} + \frac{\phi''^2}{\phi'^2} + \frac{\phi''}{\phi'} - (n-1)\frac{\phi''}{\phi} + (n-1)\frac{\phi'^2}{\phi^2} \\ & +(n-1)\frac{\phi'}{\phi} - n\end{aligned} \tag{4.22}$$

such that the Ricci tensor has the form

$$R_{i\bar{j}} = \alpha(s)e^{-s}\delta_{ij} + \beta(s)e^{-s}\delta_{ij1} \tag{4.23}$$

The expression for Ricci scalar R can also be described.

Lemma 3. *The Ricci scalar R for the rotationally Kähler metric $g_{i\bar{j}}$ is given by*

$$R = -\frac{\phi'''}{\phi'^2} + \frac{\phi''^2}{\phi'^3} - 2(n-1)\frac{\phi''}{\phi'\phi} - (n-1)(n-2)\frac{\phi'}{\phi^2} + \frac{n(n-1)}{\phi} \tag{4.24}$$

Proof. First, we recall that for any rotationally invariant Kähler metric on $\mathbb{C}^n$, the Ricci tensor takes the form

$$R_{i\bar{j}} = \alpha(s)e^{-s}\delta_{ij} + \beta(s)e^{-2s}\bar{z}_i z_j \tag{4.25}$$

where $\alpha(s)$ and $\beta(s)$ are described in equations (4.21) and (4.22). Then we can calculate as follows,

$$\begin{aligned}R &= g^{i\bar{j}}R_{i\bar{j}} \\ &= \Big(\frac{1}{\phi}e^s\delta^{ij} + \Big(\frac{1}{\phi'} - \frac{1}{\phi}\Big)z^i\bar{z}^j\Big)\Big(\alpha(s)e^{-s}\delta_{ij} + \beta(s)e^{-2s}\bar{z}_i z_j\Big) \\ &= \frac{n-1}{\phi}\alpha(s) + \frac{1}{\phi'}(\alpha(s)+\beta(s)) \\ &= \frac{n-1}{\phi}\Big(-\frac{\phi''}{\phi'} - (n-1)\frac{\phi'}{\phi} + n\Big) + \frac{1}{\phi'}\Big(-\frac{\phi''}{\phi'} - (n-1)\frac{\phi'}{\phi} + n \\ &\quad -\frac{\phi'''}{\phi'} + \frac{\phi''^2}{\phi'^2} + \frac{\phi''}{\phi'} - (n-1)\frac{\phi''}{\phi} + (n-1)\frac{\phi'^2}{\phi^2} + (n-1)\frac{\phi'}{\phi} - n\Big) \\ &= -\frac{\phi'''}{\phi'^2} + \frac{\phi''^2}{\phi'^3} - 2(n-1)\frac{\phi''}{\phi'\phi} - (n-1)(n-2)\frac{\phi'}{\phi^2} + \frac{n(n-1)}{\phi}\end{aligned}$$

and it is the statement for Ricci scalar that we want. □

4.2 The Curvatures of Rotationally Invariant Gradient Kähler-Ricci Soliton

After we have succeeded to write the expressions for Ricci tensor and Ricci scalar of any rotationally invariant Kähler metric in lemma 2 and 3, we want to get the corresponding expressions for the rotationally invariant gradient Kähler-Ricci soliton. In fact, we have

Theorem 4. *The Ricci curvature tensor $R_{i\bar{j}}$ and Ricci scalar R of the rotationally invariant gradient Kähler-Ricci soliton $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u(s)$, where $s \equiv \ln |z|^2$ and $u' \equiv \phi$, are*

$$\begin{aligned} R_{i\bar{j}} &= -(\mu\phi' + \lambda\phi)e^{-s}\delta_{ij} + (-\mu\phi'' + (\mu - \lambda)\phi' + \lambda\phi)e^{-2s}\bar{z}_i z_j \quad (4.26) \\ R &= -\mu^2\phi' - \mu\lambda\phi - (\mu + \lambda)n \quad (4.27) \end{aligned}$$

Proof. Recall that the rotationally invariant gradient Kähler-Ricci soliton satisfies equation (3.28), then for this soliton, the functions $\alpha(s)$ and $\beta(s)$ take the form

$$\begin{aligned} \alpha(s) &= -\mu\phi' - \lambda\phi \quad (4.28) \\ \beta(s) &= -\mu\phi'' + (\mu - \lambda)\phi' + \lambda\phi \quad (4.29) \end{aligned}$$

where to get the expression for $\beta(s)$ above we have differentiate equation (3.28) with respect to s to get the equation

$$\frac{\phi'''}{\phi'} - \frac{\phi''^2}{\phi'^2} + (n-1)\Big(\frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2}\Big) = \mu\phi'' + \lambda\phi' \quad (4.30)$$

Then by using the expressions for $\alpha(s)$ and $\beta(s)$ above, equation (4.14) for the Ricci tensor now becomes

$$R_{i\bar{j}} = -(\mu\phi' + \lambda\phi)e^{-s}\delta_{ij} + (-\mu\phi'' + (\mu - \lambda)\phi' + \lambda\phi)e^{-2s}\bar{z}_i z_j \quad (4.31)$$

To find the form of Ricci scalar, we should use equations (4.28) and (4.29).

$$\begin{aligned}
R &= \frac{n-1}{\phi}\alpha(s) + \frac{1}{\phi'}(\alpha(s)+\beta(s)) \\
&= -(n-1)\frac{1}{\phi}(\mu\phi' + \lambda\phi) - \frac{1}{\phi'}(\mu\phi'' + \lambda\phi') \\
&= -\mu\Big(\frac{\phi''}{\phi'} + (n-1)\frac{\phi'}{\phi}\Big) - \lambda n \\
&= -\mu(\mu\phi' + \lambda\phi + n) - \lambda n \\
&= -\mu^2\phi' - \mu\lambda\phi - (\mu+\lambda)n \qquad (4.32)
\end{aligned}$$

where we have used the equation (3.28) to simplify things. □

Remark. It is important to note that for $\mu = 0$ and $\lambda \neq 0$, the Ricci tensor and Ricci scalar have the forms $R_{i\bar{j}} = -\lambda g_{i\bar{j}}$ and $R = -\lambda n$, for which the Kähler metric $g_{i\bar{j}}$ becomes Kähler-Einstein metric. And for if $\mu = \lambda = 0$, the metric is Ricci flat, because $R_{i\bar{j}}$ and R vanish.

4.3 Equivalency of the Constant Ricci Scalar and the Kähler-Einstenian Notion

By using the results of our previous calculation, we can establish this theorem.

Theorem 5. *Suppose we have a rotationally invariant gradient Kähler-Ricci soliton $g_{i\bar{j}}$ on the complex plane $\mathbb{C}^n$, with soliton parameter λ and the holomorphic vector $V^i = -\mu z^i$ which generates the biholomorphism, then*
Case $\lambda \neq 0$, $\mu = 0$. All solitons have constant nonzero Ricci scalar and are Kähler-Einstein.
Case $\lambda = 0$, $\mu \neq 0$. No soliton has constant Ricci scalar and is Kähler-Einstein.
Case $\lambda = 0$, $\mu = 0$. All solitons have vanishing Ricci scalar and Ricci tensor.
Case $\lambda \neq 0$, $\mu \neq 0$. Both constancy of Ricci scalar and Kähler-Einsteinian case happen when $\mu = -\lambda$. If either happens, both will vanish and the soliton is trivial.

Proof. Recall that the Ricci scalar and Ricci tensor have the form

$$R = -\mu^2\phi' - \mu\lambda\phi - (\mu+\lambda)n \tag{4.33}$$
$$R_{i\bar{j}} = -(\mu\phi' + \lambda\phi)e^{-s}\delta_{ij} + (-\mu\phi'' + (\mu-\lambda)\phi' + \lambda\phi)e^{-2s}\bar{z}_i z_j \tag{4.34}$$

and the soliton on $\mathbb{C}^n$ must satisfies the equation

$$\frac{\phi''}{\phi'} + \Big(\frac{n-1}{\phi} - \mu\Big)\phi' = n + \lambda\phi \tag{4.35}$$

If R is constant, say $R = \kappa n$ for a constant κ, then this equation holds

$$\mu^2\phi' + \mu\lambda\phi + (\mu+\lambda+\kappa)n = 0 \tag{4.36}$$

and if $g_{i\bar{j}}$ is Kähler-Einstein, say $R_{i\bar{j}} = \nu g_{i\bar{j}}$ for a constant ν, then these equations hold

$$\mu\phi' + (\lambda+\nu)\phi = 0 \tag{4.37}$$
$$-\mu\phi'' + (\mu - (\lambda+\nu))\phi' + (\lambda+\nu)\phi = 0 \tag{4.38}$$

which come from the independency of δ_{ij} and $\bar{z}_i z_j$.

Case $\lambda \neq 0$, $\mu = 0$.

From (4.33) and (4.34), we have that $R = -\lambda n$ and $R_{i\bar{j}} = -\lambda g_{i\bar{j}}$. These results also come from equations (4.36), (4.37) and (4.38) which yield $\kappa = \nu = -\lambda$.

Case $\lambda = 0$, $\mu \neq 0$.

From equation (4.36), we have

$$\mu^2\phi' + (\mu+\kappa)n = 0 \tag{4.39}$$

which has the solution $\phi = -\big(\frac{\mu+\kappa}{\mu^2}\big)ns + c$, for a constant c. Then we have $\phi' = -\big(\frac{\mu+\kappa}{\mu^2}\big)n$ and $\phi'' = 0$. Then substituting this into equation (4.35), then we have

$$-\Big(\frac{\mu+\kappa}{\mu^2}\Big)ns + c - \frac{n(n-1)\big(\frac{\mu+\kappa}{\mu^2}\big)}{\frac{\kappa}{\mu}n} = 0 \tag{4.40}$$

which implies $\mu + \kappa = 0$ and $c = 0$, and hence $\phi = 0$, which contradicts the positive-definiteness of Kähler metric.

From equations (4.37) and (4.38), we have $\phi = ce^{-\nu s/\mu}$, for a constant c. After substituting this into equation (4.35), we have

$$\nu ce^{-\nu s/\mu} - \Big(\frac{\mu+\nu}{\mu}\Big)n = 0 \tag{4.41}$$

which means $\nu = 0$ and hence $n = 0$, or for the second possibility it implies that $\mu + \nu = 0$ and $\nu = 0$, which means that $\mu = 0$, where either cases lead to a contradiction.

Case $\lambda = \mu = 0$.

From equations (4.33) and (4.34) we have $R = 0$ and $R_{i\bar{j}} = 0$. A remark in the previous section also proves this case.

Case $\lambda \neq 0$, $\mu \neq 0$.

From equation (4.36) we have

$$\phi' + \frac{\lambda}{\mu}\phi + \Big(\frac{\mu+\lambda+\kappa}{\mu^2}\Big)n = 0 \tag{4.42}$$

which has the solution

$$\phi = -\Big(\frac{\mu+\lambda+\kappa}{\mu\lambda}\Big)n + ce^{-\lambda s/\mu} \tag{4.43}$$

for a constant c. After subtituting this into equation (4.35), we have

$$-\frac{1}{c}\Big(\frac{\mu+\lambda+\kappa}{\mu\lambda}\Big)ne^{\lambda s/\mu} + 1 - \frac{(n-1)\frac{\lambda}{\mu}}{(n-1)\frac{\lambda}{\mu} + n\frac{\kappa}{\mu}} = 0 \tag{4.44}$$

which implies $\mu + \lambda + \kappa = 0$ and $\kappa = 0$. Hence $\mu = -\lambda$, and the soliton $g_{i\bar{j}}$ is identity metric, unique up to scaling. Since $\kappa = 0$, then $R = 0$.

From equations (4.37) and (4.38) we have

$$\begin{aligned} \mu\phi' + (\lambda+\nu)\phi &= 0 \qquad (4.45)\\ -\mu\phi'' + (\mu - (\lambda+\nu))\phi' + (\lambda+\nu)\phi &= 0 \qquad (4.46) \end{aligned}$$

The solution of these two equations is either $\phi = ce^{s}$ or $\phi = ce^{-(\lambda+\nu)s/\mu}$, for a constant c. Substituting the latter to equation (4.35) we will get

$$\nu ce^{-(\lambda+\nu)s/\mu} - \Big(\frac{\mu+\nu+\lambda}{\mu}\Big)n = 0 \tag{4.47}$$

then $\nu = 0$ and $\mu + \nu + \lambda = 0$, which yield $\mu = -\lambda$, and the soliton $g_{i\bar{j}}$ is identity metric, unique up to scaling. Since $\nu = 0$, then $R_{i\bar{j}} = 0$. □

From this theorem we have this

Corollary 6. *On the complex plane $\mathbb{C}^n$, the rotationally invariant gradient Kähler-Ricci soliton $g_{i\bar{j}}$ is Kähler-Einstein if and only if its Ricci scalar is constant.*

Chapter 5

Conclusion

From theorem 5, or more explicitly in corollary 6, we conclude that the constancy of Ricci scalar curvature and the Kähler-Einsteinian notion are completely equivalent for the case of rotationally invariant gradient Kähler-Ricci solitons. This equivalence is not trivial in the sense that it generally does not hold for any Kähler metric, so it makes the structure of solitons in rotationally symmetric setup much more interesting. We also conclude that in the case where the Ricci scalar could be not constant or the soliton could be not Kähler-Einstein, if we set either Ricci scalar to be constant or the soliton is Kähler-Einstein, then they must vanish identically, and this condition could be only achieved in the case where $\mu = -\lambda$.

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Appendix A

Einstein Summation Convention

Throughout the text I assume the *Einstein summation convention* holds, i.e. if we have a quantity which has one subscript index and one same superscript index, then it means that we must add over the possible values of the index. For example, if we have a quantity

$$U_\mu U^\mu \tag{A.1}$$

where μ runs from 1 to n, then the quantity is defined as

$$U_\mu U^\mu \equiv U_1 U^1 + \cdots + U_n U^n = \sum_{\mu=1}^{n} U_\mu U^\mu \tag{A.2}$$

Another example is

$$U_\mu T^{\mu\nu} \equiv U_1 T^{1\nu} + \cdots + U_n T^{n\nu} \tag{A.3}$$

In equation (A.2), the quantity $U_\mu U^\mu$ is a scalar, a real number, since it comes from the summation of the corresponding components of U_μ and U^μ. But in equation (A.3), $U_\mu T^{\mu\nu}$ is a vector, and the equation itself is indeed n equations embedded in a single one.

Appendix B

Manifolds

B.1 Introduction

When I was a child I often asked a question about the best way to measure the distance over the curved object, such as mountain and hill.

Naturally, the answer is so easy. The first idea that came to my mind is about measuring the length of a thread connecting two points in that curved surface while keeping this thread along the surface. With this method we still get the intuitive notion that distance is the smallest possible length of line that connects those two points. But technically, it is very difficult to do; it seems there is no standard way to make it easy, unlike in the flat surface[1]. This difficulty arises from the fact that our surrounding will appear different when we are in different points in the curved surface. This obviously does not happen in the flat one, because if we stand in any points and see to every directions, it appears identically: an empty plane. Due to its simplicity, we will observe what we can do with the flat space[2] (usually it is denoted as $\mathbb{R}^n$, where n is the space dimension, and it is also known as the *Euclidean space*) first before we tackle the curved one.

When we talk about the flat space (and any other ones), we can think that this is the background or place for something to happen, or see this

[1]Here, we also can use the "thread method" to find a distance between two points, but due to its flatness, we can simply mark each points on the surface with a tuple of numbers, and describe the distance by using these numbers.

[2]I will use the term flat space as the flat surface of any arbitrary dimension other than 2 as well.

object in its entirety and regard it as the main focus of our attention without concerning about something else. We should use the second view. When we are looking to a flat space, we can see that it is nothing other than a collection of many points, or elements. We will need to parameterize these points such that we can refer to it easily. Therefore, the natural way to identify the points are to relate them with some easier objects that we can encounter, i.e. the n-tuple of numbers. Given a point p which belongs to the space $\mathbb{R}^n$, we can know where the point is by using the n-tuple of numbers for this point. The mapping which relates the points on the flat space $\mathbb{R}^n$ to their n-tuple of numbers is called the *coordinate system.*

It is natural to make a requirement that our coordinate system must specify the points in $\mathbb{R}^n$ to the n-tuple of numbers *uniquely* and each points should have *single-valued* tuple. In $\mathbb{R}^n$, it is possible to construct such coordinate system. If the coordinate of a point is denoted as $\{x^\mu\}$, where $\mu = 1, \ldots, n$, then we will know that a different number for each μ in this coordinate means different points in $\mathbb{R}^n$.

However, the structure of space and some physical events which occur in this flat space will not be affected by our choice of coordinate system. Whether we choose the Cartesian or spherical or cylindrical system, nature does not know. Therefore, it is an advantage for us to choose the simplest possible system, which is usually the orthogonal one such as Cartesian, although it is not an absolute principle.

Because the choice of coordinate system to identify the points in $\mathbb{R}^n$ is not unique, then we can make many systems in this space. By principle, they are equally likely each other to mark the elements of flat space, since nature does not provide the way to distinguish them. Therefore, we can construct the coordinate systems as many as we can, and it means that we must provide a way to transfer the information obtained from one system to another. This way is called the *coordinate transformation.* Given some coordinate systems and how to transform the coordinates between them, we can know the values of some physical quantities in another frame if we know their values in at least one coordinate system. For the physical situation, if there are many observers who see the same event, then the agreement between them only comes after the coordinate transformations.

By using these principles, that we can make the coordinate system to identify the points in the flat Euclidean space and the existence of a consistent transformation between the coordinate systems, will help us much to construct the theory for the curved space.

B.2 Definition

What we can do in the flat Euclidean space cannot be generalized to the curved one, because by nature, the surroundings of every points in the curved space are different. We cannot make an identification which is as easy as in the flat one, but fortunately we can do it in a local area in the curved space. It is possible because if we make any small area in the curved space, the situation will appear like we are in the flat one; the fact which is motivated highly by our daily intuitive notion about the flatness of Earth in our region although it is indeed a sphere. Therefore, it is an advantage for us to highlight this property of local flatness, and define the curved space which[3] locally looks like the Euclidean space $\mathbb{R}^n$ as the *n-dimensional manifold*.

Mathematically, it means that for every point $p \in M$, we can make a neighborhood $U \subset M$ which contains p (i.e. $p \in U$) such that locally in this neighborhood U there exists a mapping $\phi : U \to \mathbb{R}^n$ which maps the neighborhood U to the flat space $\mathbb{R}^n$. The existence of ϕ guarantees that the manifold M locally looks like the $\mathbb{R}^n$. Therefore, what we can do in the previous section for the Euclidean space can also be done here in $\mathbb{R}^n$ which mimics the manifold M only locally.

The introduction of the local mapping ϕ in our theory to identify the points in M by a local coordinate system is very important, in the sense that we cannot make a single coordinate system which works globally over the manifold. If we insist to make this one, in some cases we will have the

[3]If we view the curved space as the collection of points, then there are numerous numbers of curved space which cannot be approximated as the flat one locally in each its points.

coordinate which is not continuous, and in other cases we will have points in M which have multiple values of coordinate. It is very hard to work in both situations, so it is best if we introduce many coordinate systems in every neighborhoods of M such that every two nearby points[4] have nearby values of coordinate, and every points have a single value of coordinate in at least one local coordinate system.

The problem of coordinate transformation in the flat space can also be formalized in the manifold. Since we can make any coordinate systems we want in $\mathbb{R}^n$, it means that we can also make any coordinate systems in each local neighborhood of M. If we have two neighborhoods $U, V \subset M$ we have two mappings which map these two neighborhoods to $\mathbb{R}^n$, i.e.

$$\phi : U \to \mathbb{R}^n, \qquad \psi : V \to \mathbb{R}^n \tag{B.1}$$

If $U \cap V \neq \emptyset$, then every points $p \in U \cap V$ will have two coordinate values defined by ϕ and ψ, and the coordinate transformation between them is represented by the composition mapping

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \tag{B.2}$$

For the sake of consistency, we require that this composition mapping $\psi \circ \phi^{-1}$ must be continuous for every pairs U and V in the manifold M, and has continuous inverse $\phi \circ \psi^{-1}$. If the coordinate in $\phi(U)$ and $\psi(V)$ has the form $\{x^\mu\}$ and $\{y^\mu\}$ respectively, then explicitly the coordinate transformation between them is

$$y^\mu = \frac{\partial y^\mu}{\partial x^\nu} x^\nu, \qquad \text{for } \mu = 1, 2, \ldots, n \tag{B.3}$$

where we assume that Einstein summation convention holds in this equation (and any other equations that appear in this skripsi. See appendix A.). If this transformation and its inverse are differentiable for every pairs U and V in M, then M is called *differentiable manifold.*

[4]Formally, I mean the two points that are in the same neighborhood.

B.3 Some Examples

Here are some simple manifolds which, apart from their simplicity, have rich structures and undeniably will increase our understanding about the concept of manifold. I pick these examples from [15].

Example 1. The simplest manifold is the Euclidean space itself, where the mapping ϕ is the identity, and this mapping holds for all parts in this space. It is equivalent to say that the neighborhood U in which the mapping ϕ can be applied is the entire $\mathbb{R}^n$.

Example 2. The only 1-dimensional manifold M which is connected is the circle S^1. The natural coordinate system in this circle is by giving the one parameter usually called as the angle θ. By setting $\theta = 0$ to a point in the circle, say p, then the value of θ increases as we move along the circle in counterclockwise direction by a usual convention. Continuing our motion, we will arrive at points close to our original point p, and which we will be asked a question, whether we want to identify that original point p as its original angle 0, or 2π since we have moved along a complete circle. If we choose 0, then our coordinate system is discontinuous, with a jumping from $2\pi - \varepsilon$, for $\varepsilon \ll$ to 0. And if we choose the latter, we will have the coordinate system which is not unique; all points will have coordinate values $\{\theta + 2k\pi\}$, for $k \in \mathbb{N}$.

One way out is to introduce two neighborhoods $U, V \subset S^1$ where U covers the upper half of this circle and V is for the lower half. By using these neighborhoods, we have two mappings $\phi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$ which, for example, have the form

$$\phi \;:\; U \to (-\varepsilon, \pi + \varepsilon) \tag{B.4}$$

$$\psi \;:\; V \to (\pi - \varepsilon, \varepsilon) \tag{B.5}$$

where the coordinate transformation $\psi \circ \phi^{-1}$ occurs in the intersection of U and V.

Example 3. The n-dimensional sphere S^n is one of example of manifold, and it is differentiable. We must face the same problem we meet in the case of circle S^1 when we want to make a single global coordinate system in S^n. But we also can make two neighborhoods, each occupies the upper and lower hemisphere of S^n, such that each neighborhoods are mapped to the $\mathbb{R}^n$.

Example 4. The n-dimensional *real projective space* $\mathbb{R}P^n$ is defined as the quotient space $(\mathbb{R}^{n+1} - \{0\})/\sim$, where $\sim$ is the equivalence relation defined in $\mathbb{R}^{n+1} - \{0\}$, i.e. for $x, y \in \mathbb{R}^{n+1} - \{0\}$ then $x \sim y$ means there exists $0 \neq k \in \mathbb{R}$ such that $x = ky$. It means that all points which lie on the same line through the origin of $\mathbb{R}^{n+1}$ can be considered as the same element in $\mathbb{R}P^n$.

Since the coordinate of points is described by $n+1$ numbers $x_0, \ldots, x_n$ then it can be used for points in the $\mathbb{R}P^n$. By introducing the neighborhood U_i where it contains the points with $x_i \neq 0$ for some i, then the coordinate of points in U_i can be described as

$$\{\chi_1, \ldots, \chi_n\} = \{x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i\}$$

and it explains why our projective space has n dimensions. Therefore the mapping $\phi_i : U_i \to \mathbb{R}^n$ has the form

$$\phi_i(\{x_0, \ldots, x_n\}) = \{\chi_1, \ldots, \chi_n\}$$

If a point $p \in U_i \cap U_j$ then there exist two mappings ϕ_i and ϕ_j for each neighborhoods,

$$\phi_i(\{x_0, \ldots, x_n\}) = \{\chi_1, \ldots, \chi_n\}, \qquad \phi_j(\{x_0, \ldots, x_n\}) = \{\eta_1, \ldots, \eta_n\}$$

where

$$\{\eta_1, \ldots, \eta_n\} = \{x_0/x_j, \ldots, x_{j-1}/x_j, x_{j+1}/x_j, \ldots, x_n/x_j\}$$

Therefore, the coordinate transformation between them is

$$(\phi_j \circ \phi_i^{-1})(\chi_k) = \frac{x_j}{x_i}\eta_k$$

Example 5. *Grassmanian manifold* $G_{k,n}(\mathbb{R})$ is defined as the k-dimensional hyperplane in $\mathbb{R}^n$. Therefore, it is a generalization of the projective space $\mathbb{R}P^n$, i.e. $\mathbb{R}P^n = G_{1,n+1}(\mathbb{R})$.

Example 6. *Torus* can be defined as a result of the compactification of a square in $\mathbb{R}^2$, but it also can be viewed as the product of the circle S^1 with itself, i.e. $T = S^1 \times S^1$. As the result, torus can be described by two parameters, each for these two circles, and hence it also can be viewed as the manifold in $\mathbb{R}^4$, in which it is a flat manifold in this space, apart from our daily imagination that torus / doughnut is a curved manifold in $\mathbb{R}^3$.

B.4 Curves and Functions

Since the manifold is the generalization for the flat space, then consequently we need to make a proper generalization for every concepts in the flat space such that we can use them, if possible, when we are working in the manifold. In this subsection we will discuss about the curve and function defined over the manifold, and in the next subsection we will discuss about some important and useful objects, i.e. vector, dual vector and tensor fields.

In the flat plane $\mathbb{R}^n$, we often analyzed the track of a particle that moves under some potential. Ordinarily, the curve on the plane can be parameterized using one parameter, say t, such that every points on the curve can be marked with some definite value of this parameter. We can also make an arbitrary choice to place the zero of t; nature will not know which point you identify with the $t = 0$. If I make a coordinate system on this plane, then a point on the curve has the coordinate

$$(x^1, \ldots, x^n) = (x^1(t), \ldots, x^n(t))$$

such that if you have a single number t, then you will have n numbers, i.e. the x^μs (for $1 \leq \mu \leq n$), and it gives you the precise position of the point on the curve (this is why we often say that curve is one-dimensional). You can make an arbitrary curve on this plane which touches every parts of this plane, but by principal, you can always identify its points using this single parameter t.

However, this advantage to use the powerful parameter t to mark points on the curve cannot hold peacefully if we make a generalization from the plane to a manifold. The problem is, we cannot make a global coordinate system which covers all parts of the manifold. Instead, we need to define some local coordinate systems, provided that we still can transform each other using continuous mapping. Therefore, the principal that we can identify the points on the curve by using coordinate $\{x^\mu\}$ doesn't make sense anymore, since two neighborhoods on the manifold have different convention about the values of x^μs.

To solve this problem, we need to make two mappings. First, consider the mapping $c : [a, b] \in \mathbb{R} \to M$ such that $c(s) = c(t)$ whenever $s = t \in \mathbb{R}$. We can imagine that this is the curve which connects two points on the manifold, namely $p = c(a)$ and $q = c(b)$, and for the sake of simplicity we make an assumption that it is a simple curve; there is no intersection of this curve with itself. Of course you don't need to have finite numbers for a and b, since the two queens of pain ($-\infty$ and ∞) will also work. While the first mapping is the one which maps $\mathbb{R}$ to M, the second mapping is $\phi : U \to \mathbb{R}^n$, i.e. the mapping that serves as the coordinate marker for each points on some neighborhood in M. Therefore, we can construct the composition mapping $\phi \circ c : [a, b] \to \mathbb{R}^n : t \mapsto \{x^1, \ldots, x^n\}$, such that from a single parameter t you will get n values of coordinates, in a neighborhood U.

If the part of $\mathbb{R}$ is mapped to M and produces what we intuitively call as curve, then the mapping from M to $\mathbb{R}$ makes a very different object: this is simply the ordinary function defined on a manifold[5]. Concretely, if we have a function $f : M \to \mathbb{R}$, and take a point $p \in M$ with the coordinate $\phi(p)$, then we will have the composition mapping $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$. In physics, we often encounter this composition mapping $f \circ \phi^{-1}$, which relates the coordinates of points on the space with their values in real numbers. Actually, this physicist's view will not make any trouble if we can define a global coordinate system on the manifold, but if we just could make a system of local coordinates, then we need to distinguish the role of f, the function from points of the manifold to some real numbers, and the role of ϕ, the attachment of points to a local coordinate system.

[5]We will denote $\mathcal{F}(M)$ as the set of functions defined over the manifold M, i.e. $\mathcal{F}(M) = \{f \mid f : M \to \mathbb{R}\}$

B.5 Vector, Dual Vector, and Tensor Fields

Vector field in a certain space can be described as the mapping from the points to a vector. In flat space, we can state the vector field easily in terms of basis vectors, the set of linear independent vectors which span that space. If our coordinate is Cartesian, as an example, we can construct the n orthonormal vectors[6] which are linear independently each other and span the space, usually denoted as e_μ, which work globally over the space. The decision to make an orthonormal system is very optional, and in fact for the flat space we can make any set of vectors which satisfy the conditions above as the basis vectors. Unfortunately, we cannot use this method in a manifold, but we can use the analogy from the expansion of function using Taylor series to define the local basis vectors [9].

Suppose we have an n-dimensional manifold M, and a neighborhood $U \subset M$ which contains $p \in M$. Then, consider another point $q \in U$ near p such that the coordinate of q differs with p by a small number ε^μ, i.e.

$$y^\mu = x^\mu + \varepsilon^\mu \tag{B.6}$$

where $\{y^\mu\}$ and $\{x^\mu\}$ are the coordinates of q and p respectively. Then, if we have a function f defined over the manifold, $f : M \to \mathbb{R}$, and if the value of this function at p is $f(p)$, then its value at q can be expanded using Taylor series as

$$f(q) = f(p) + \varepsilon^\mu \frac{\partial f}{\partial x^\mu}\bigg|_p + \cdots \tag{B.7}$$

From equation (B.7) above, we can see that the difference between $f(q)$ and $f(p)$ in the first order is $\varepsilon^\mu \frac{\partial f}{\partial x^\mu}$, the *directional derivative* of the function f (i.e. the derivative of f on the direction along the curve which connects p and q). We can guess that this directional derivative does not depend on the coordinate system we use; for if we are given a function f on a manifold M, the value of directional derivative of f at a certain point on M is coordinate independent. Then, we can also guess that

$$\varepsilon^\mu \frac{\partial}{\partial x^\mu} \tag{B.8}$$

[6] It is obvious that the set of vectors which span the flat space and are linear independent must contain n elements.

is the tangential vector of the curve which connects points p and q, by setting $\lim \varepsilon \to 0$.

Being the vector, we can view the ε^μ in equation (B.8) as its component, and $\partial/\partial x^\mu$ as the basis. This choice of basis vectors is called the coordinate basis, since it depends on the orientation of axis of coordinate system we make (remember that we can construct the basis vectors independently without noticing the axis orientation of the coordinate system being used). Therefore, we have had the basis vectors we want in manifold; the basis which will replace the orthonormal basis vectors in Cartesian system in flat space.

One thing to remember for this construction of vector is about the comparison of two vectors. On the plane, we can say that two vectors on two different points are identic if they are parallel and their components are the same. On manifold, the first statement still holds (that two vectors are the same if their directions are the same), but the second statement fails. We cannot compare the components of two vectors on two different points because the coordinate systems we use are generally different at those points. Although the vectors are parallel, their components are different. Mathematically, if we have two parallel vectors $A = A^\mu \frac{\partial}{\partial x^\mu}$ at $a \in M$ and $B = B^\mu \frac{\partial}{\partial y^\mu}$ at $b \in M$, then

$$A = A^\mu \frac{\partial}{\partial x^\mu} = B^\mu \frac{\partial}{\partial y^\mu} = B \tag{B.9}$$

but generally $A^\mu \neq B^\mu$. Since the transformation between these two basis vectors is

$$\frac{\partial}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial}{\partial x^\nu} \tag{B.10}$$

then the equation which relates the components of A and B is

$$A^\nu = \frac{\partial x^\nu}{\partial y^\mu} B^\mu \tag{B.11}$$

The two equations, (B.9) and (B.11), are the mathematical version of two statements I said previously.

Take a specific point $p \in M$. If there is a smooth curve through p, then there is a vector tangential to this curve at p. Therefore, if we can draw all possibilities of curves through p, we can draw all vectors which are tangential to manifold M at p. The vector space which is spanned by the set of tangential vectors at p is denoted as T_pM, where the subcscript p is useful to stress that the different points have different tangential vector spaces[7]. Namely, the tangential vector space T_qM at q is generally different with T_pM. They are same only if our manifold is flat. The set of vector fields in M will be denoted as $\chi(M)$, such that if $X \in \chi(M)$, then the vector $X|_p \in X$ is the element of T_pM.

We also can create another object on manifold which we will name as the dual vector. Shortly, the dual vector (or, people often call it as one-form) ω can be defined as the mapping that takes a vector V into a real number $\omega(V)$. The dual vector ω can be stated explicitly as

$$\omega = \omega_\mu dx^\mu \tag{B.12}$$

where the ω^μ is the components of ω and dx^μ is its basis. The advantage of making the basis in this way is because the statement that ω is a mapping from vector to a real number can be translated to a statement that there exists inner product between the dual vector and vector in T_p^*M and T_pM respectively, which has a real number as the result, i.e.

$$\langle\omega, V\rangle = \omega(V) \in \mathbb{R} \tag{B.13}$$

By using the coordinate representation, we will have

$$\begin{aligned} \langle\omega, V\rangle &= \langle\omega_\mu dx^\mu, V^\nu \frac{\partial}{\partial x^\nu}\rangle = \omega_\mu V^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle \\ &\equiv \omega_\mu V^\nu \frac{\partial x^\mu}{\partial x^\nu} = \omega_\mu V^\nu \delta^\mu_\nu \qquad \text{(B.14)} \\ &= \omega_\mu V^\mu \in \mathbb{R} \qquad \text{(B.15)} \end{aligned}$$

The space which is spanned by the dual vectors at p is denoted as T_p^*M, and it is called the dual space of T_pM. And the set of dual vector fields in M is denoted as T^*M. Please note that the inner product we defined above is between the dual vector and the vector, not between two vectors. Don't

[7]Notice that although both T_pM and $\mathbb{R}^n = \phi^{-1}(U)$ are the "planes" tangential to manifold M, they are by definition very different in structures. T_pM, being the vector space, contains the vectors as its elements, but $\phi^{-1}(U)$ contains points because it is a flat manifold.

be trapped by the ordinary inner product between two vectors; this ordinary inner product actually can give us the length of vector. Here, there is no such definition for the length of something and distance between two points. What we only have is the components of the vector and not its length. The discussion about the length of vector and the distance between two points on the manifold is appropriate only if we have a metric on the manifold (see section C).

Now, after we successfully construct the vectors and dual vectors, we can make a straightforward generalization to an object called tensor. If we define the dual vector as a mapping from vector to real number, then we define the tensor of type (r, s) as a mapping from r dual vectors and s vectors to a real number. Then, we can say that the vector is a tensor of type $(1, 0)$ (because if we have a vector, then we need one dual vector to make a real number) and the dual vector is a tensor of type $(0, 1)$. If the coordinate representation of vector V is $V = V^\mu \frac{\partial}{\partial x^\mu}$ and dual vector ω is $\omega = \omega_\mu dx^\mu$, then the (r, s)-tensor T is represented as[8]

$$T = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \bigotimes_{i=1}^{r} \frac{\partial}{\partial x^{\mu_i}} \bigotimes_{j=1}^{s} dx^{\nu_j} \tag{B.16}$$

such that if we have r dual vectors $\omega_1, \ldots, \omega_r$ and s vectors $V_1, \ldots, V_s$, the mapping T is

$$T(\omega_1, \ldots, \omega_r, V_1, \ldots V_s) = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \omega_{1\mu_1} \ldots \omega_{r\mu_r} V_1{}^{\nu_1} \ldots V_s{}^{\nu_s} \tag{B.17}$$

For the next discussion we will denote the tangential (r, s)-tensor space at a point $p \in M$ as $\mathscr{T}^r_{s,p} M$, and the set of (r, s)-tensor fields in manifold M as $\mathscr{T}_s{}^r M$.

The coordinate transformation between components of tensor is also easy to develop. If we have a tensor T on a manifold M, and we make two different coordinate systems $\{x^\mu\}$ and $\{y^\mu\}$ on the neighborhood U of point p, then

$$T = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \bigotimes_{i=1}^{r} \frac{\partial}{\partial x^{\mu_i}} \bigotimes_{j=1}^{s} dx^{\nu_j} = T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} \bigotimes_{i=1}^{r} \frac{\partial}{\partial x^{\alpha_i}} \bigotimes_{j=1}^{s} dx^{\beta_j} \tag{B.18}$$

[8]Actually this representation is called as Einstein-Penrose representation for tensor. There is another representation, a graphical one, made by Penrose [16] which is useful for classification of Lie groups. This is far out of our topic.

such that the transformation between the components of tensor T is

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = \prod_{i=1}^{r}\prod_{j=1}^{s}\frac{\partial x^{\mu_i}}{\partial y^{\alpha_i}}\frac{\partial y^{\beta_j}}{\partial x^{\nu_j}}T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s} \tag{B.19}$$

Any indexed object on the manifold which transforms like the equation (B.19) above is a tensor. It's why we commonly use the coordinate transformation, which is simple and easy to use, as the defining method to distinguish a tensor, not its original definition as a mapping from a number of vectors and a number of dual vectors to a real number.

B.6 Induced Mapping

Suppose we have a mapping $f : M \to N$ which maps a point from manifold M to N, then naturally it will induce the mapping f_* which is called the *differential mapping*, from the tangential vector space of p in M to the tangential vector space of $f(p)$ in N, i.e.

$$f_* : T_pM \to T_{f(p)}N \tag{B.20}$$

Given a vector $V \in T_pM$, then the vector $f_*V \in T_{f(p)}N$ is defined as

$$f_*V[g] = V[g \circ f] \tag{B.21}$$

for a function $g : N \to \mathbb{R}$. If we denote $V = V^\mu\partial/\partial x^\mu$ and $f_*V = W^\nu\partial/\partial y^\nu$ for the coordinate $\{x^\mu\}$ of p and $\{y^\nu\}$ of $f(p)$, then from equation (B.21) above we will have

$$W^\nu\frac{\partial}{\partial y^\nu}[g] = V^\mu\frac{\partial}{\partial x^\mu}[g \circ f] \tag{B.22}$$

such that if we set g to be y^ν we will have

$$W^\nu = V^\mu\frac{\partial y^\nu}{\partial x^\mu} \tag{B.23}$$

This relation can also be generalized to the case of $(r, 0)$-tensor, i.e.

$$f_* : \mathscr{T}^r_{0,p}M \to \mathscr{T}^r_{0,f(p)}N \tag{B.24}$$

such that it yields

$$S^{\nu_1\cdots\nu_r} = \prod_{i=1}^{r}\frac{\partial y^{\nu_i}}{\partial x^{\mu_i}}T^{\mu_1\cdots\mu_r} \tag{B.25}$$

for $T \in \mathscr{T}_0^r M$ and $S \in \mathscr{T}_0^r N$.

The mapping $f : M \to N$ also induces the *pullback* mapping $f^* : T^*_{f(p)}N \to T^*_p M$, defined as

$$\langle f^*\omega, V\rangle = \langle \omega, f_* V\rangle \tag{B.26}$$

where $\omega \in T^*_p N$ and $V \in T_p M$. If we define $\eta \equiv f^*\omega \in T^*_p M$, then we can calculate

$$\omega_\nu = \eta_\mu \frac{\partial x^\mu}{\partial y^\nu} \tag{B.27}$$

This also can be extended to the case of $(0, s)$-tensor, i.e.

$$f^* : \mathscr{T}^0_{s,p} M \to \mathscr{T}^0_{s,f(p)} N \tag{B.28}$$

such that it yields

$$S_{\nu_1\cdots\nu_s} = \prod_{i=1}^{s} \frac{\partial x^{\nu_i}}{\partial y^{\mu_i}} T_{\mu_1\cdots\mu_s} \tag{B.29}$$

for $T \in \mathscr{T}_s^0 M$ and $S \in \mathscr{T}_s^0 N$.

There is no natural extension of the induced mapping for the case of mixed tensor, i.e. for $\mathscr{T}_s^r M$, with $r, s \neq 0$.

B.7 Flows and Lie Derivatives

In physics we are familiar with this situation: there is an area (or volume) which has the vector field which becomes the tangential vector of some curves in this area. The trivial example is a river, where the curve is the motion track of a particular dust floating down the river, and where the tangential vector is indeed the velocity of the stream. Another situation is where you draw the lines around the magnet which point from the north pole to the south pole, where the tangential vector is the magnetic field of that magnet. Due to this familiarity in physical world, it can be an advantage if we can define the similar concept in the manifold.

Suppose we have an n-dimensional manifold with the vector field X defined over this manifold. Provided the neighborhood U and the mapping

$\phi : U \to \mathbb{R}^n$ around the point $p \in M$, we can construct the curve which has the form

$$\frac{dx^\mu(t)}{dt} = X^\mu(x(t)) \tag{B.30}$$

where $\{x^\mu(t)\}$ is nothing other than $\phi(p)$. We can see easily from equation above that the vector field X is the tangential vector of this curve, and we will call it the *integral curve* for the next discussion. There will be possibly many integral curves on the same manifold, with a given vector field. We can define the equivalence class such that two points in M belong to the same class if they are connected to the same curve. For this, we can single out one point in each class which represents that class, to make the analysis easier for the next time.

Suppose we have one of those equivalence classes, and a point x_0 as its representation. Then, the integral curve for this point is

$$\frac{d}{dt}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)) \tag{B.31}$$

where we have changed the symbol from x to σ because we want to stress a point: that our integral curve passes the point x_0. Then, we can make this relation

$$\sigma(0, x_0) = x_0 \tag{B.32}$$

where we assume that the parameter t of the curve is zero in x_0. We also have

$$\sigma(t, \sigma(s, x_0)) = \sigma(t + s, x_0) \tag{B.33}$$

which tells us about the freedom to choose the initial point of the integral curve, or to choose the representation for each equivalence classes we talked above. Given the vector field X on the manifold, we can principally find the parametric equation for the integral curve by using that flow equation.

We can change a point of view in this point. If we remember that the initial point of the integral point is free to choose, then we can assume it's undefined initially, and we can write the integral curve as $\sigma(t, x)$ which maps the parameter t and the point $x \in M$ to another point $\sigma(t, x) \in M$ in the manifold. This mapping $\sigma : \mathbb{R} \times M \to M$ forms the diffeomorphism from M

to the manifold $\tilde{M}$ which is diffeomorphic with M. To see this diffeomorphism clearly, ones usually use the symbol $\sigma_t : M \to M$, where we can see that in each parameter t there is the manifold M_t which is diffeomorphic to M and they are related by mapping $\sigma_t : M \to M$.

The concept of integral curves in the manifold can also be used to compare the vectors in different points. As we know, we cannot compare two vectors in two different points in the manifold because we cannot make a single global coordinate system. Thus, if we have a vector field Y in M, then we cannot compare the vector of Y on a point p and q, where $p \neq q$. We need a method to compare them, and fortunately provided the concept of integral curve, we have it. We can make an integral curve which connects two points p and q which we denote its tangential vector as X. Then we have these two flow equations

$$\frac{d\sigma_t^\mu(x)}{dt} = X^\mu(x(t)) \tag{B.34}$$

and

$$\frac{d\tau_t^\mu(x)}{dt} = Y^\mu(x(t)) \tag{B.35}$$

Suppose we want to compare two vectors $Y|_x$ and $Y_{x+\varepsilon X}$ in the points x and $x+\varepsilon X$ of M, respectively, where εX is nothing but the scaling of $X|_x$. Then, we can use the induced mapping to bring the $Y|_{x+\varepsilon X}$ to the point x, and we denote this translated vector as $\tilde{Y}|_x$. Therefore, the difference between $Y|_x$ and $Y_{x+\varepsilon X}$ now becomes

$$Y|_x - \tilde{Y}|_x = Y|_x - (\sigma_{-\varepsilon})_* Y|_{x+\varepsilon X} \tag{B.36}$$

where $(\sigma_{-\varepsilon})_*$ is the induced mapping. We can define the *Lie derivative* of Y along the integral curve of X in the point x by using the difference above

$$\mathcal{L}_X Y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}((\sigma_{-\varepsilon})_* Y|_{x+\varepsilon X} - Y|_x) \tag{B.37}$$

Now we want to work more explicitly, and suppose $X = X^\mu \partial/\partial x^\mu$ and $Y = Y^\mu \partial/\partial x^\mu$. The coordinate of $\sigma_\varepsilon(x)$ is $x^\mu + \varepsilon X^\mu$, then the vector $Y|_{x+\varepsilon X}$ is described as

$$Y|_{x+\varepsilon X} = Y^\mu(x^\nu + \varepsilon X^\nu) e_\mu|_{x+\varepsilon X} \tag{B.38}$$

such that we can obtain the vector $\tilde{Y}|_x$ by using the induced mapping $(\sigma_{-\varepsilon})_*$ to the vector above and get this result

$$\tilde{Y}|_x = Y|_x + \varepsilon(X^\mu(x)\partial_\mu Y^\nu(x) - Y^\mu(x)\partial_\mu X^\nu(x))e_\nu|_x + O(\varepsilon^2) \tag{B.39}$$

hence we can get the expression for $\mathcal{L}_X Y$ as

$$\mathcal{L}_X Y = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)e_\nu \tag{B.40}$$

If we define the *Lie bracket* $[X, Y]$ between two vector fields X and Y by the relation

$$[X, Y] = X[Y[f]] - Y[X[f]] \tag{B.41}$$

where $X[Y[f]]$ is the directional derivative of the $Y[f]$ along X, and where $Y[f] = Y^\mu \partial f/\partial x^\mu$. It appears easily that this definition coincides with that of Lie derivative, then we have

$$\mathcal{L}_X Y = [X, Y] \tag{B.42}$$

It is important to note that the Lie derivative of a function f, $\mathcal{L}_X f$, can be computed to get the result

$$\mathcal{L}_X f = X[f] \tag{B.43}$$

i.e. the usual directional derivative of the function f along X. For the case of (r, s)-tensors, the concept of Lie derivative can be used to each index. For if we have two tensors $T_1, T_2 \in \mathscr{T}_s^r M$, the Lie derivative of $T_1 + T_2$ is the sum of Lie derivatives of each tensors, i.e.

$$\mathcal{L}_X(T_1 + T_2) = \mathcal{L}_X T_1 + \mathcal{L}_X T_2 \tag{B.44}$$

and for the direct product of two tensors $T_1 \in \mathscr{T}_s^r M$ and $T_2 \in \mathscr{T}_{s'}^{r'} M$, we have

$$\mathcal{L}_X T_1 \otimes T_2 = (\mathcal{L}_X T_1) \otimes T_2 + T_1 \otimes (\mathcal{L}_X T_2) \tag{B.45}$$

Appendix C

Riemannian Manifolds

C.1 Metric Tensor

Now it's the time to introduce the concept of distance into our manifold. In section B, we have defined the curved space and generalized some objects which initially only exist in the flat space such as vector and dual vector fields, and regard them as the quantities which have the representations in terms of coordinate basis and hence also have the components, relative to this coordinate. However, in daily life we must tackle the space which has a distance concept within it and the concept of length of vector. The latter has a great significant functionality because often in physics we need to know the magnitude of quantity defined as the vector and dual vector fields. The development of the topics discussed in this section is useful for the construction and formulation of general relativity [21, 13].

For if we want to define a distance notion in a given manifold M, then we can simply take any two arbitrary points in M and define how much the distance between them. Of course, we can say that our manifold has a distance notion *only after* the distance of any pairs of its points has been defined. Notice that this distance is not affected by our choice of coordinate; the type of coordinate system and the way we use it won't change the distance we have defined between pairs of points. Hence, we should construct a mechanism which allows us to pretend the distance of two fixed points although we change the coordinate system of our manifold.

If we have two nearby points $p, q \in M$ which have the coordinate values

$\{x^\mu\}$ and $\{x^\mu + dx^\mu\}$, then the distance between them should be comparable to the difference between their coordinates, i.e. dx^μ, for $1 \leq \mu \leq n$. Precisely, if we make a vector $\mathbf{ds}$ which connects p and q, then we can represent it in terms of basis vectors as

$$\mathbf{ds} = dx^\mu e_\mu \tag{C.1}$$

where $e_\mu = \partial/\partial x^\mu$ is the basis vector, such that the distance ds between these points can be stated as the length of this vector, i.e.

$$ds^2 = \mathbf{ds} \cdot \mathbf{ds} = dx^\mu dx^\nu e_\mu \cdot e_\nu \tag{C.2}$$

If we define

$$g_{\mu\nu} \equiv e_\mu \cdot e_\nu \tag{C.3}$$

then we can have the convenient description for the distance of points p and q as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{C.4}$$

Since $g_{\mu\nu}$ is the inner product of two basis vectors, we can interpret it as the component of a $(0,2)$-tensor $g \in \mathscr{T}_2^0 M$ where for the next discussion will be called as the *metric tensor*, and we can write it in terms of basis of the vector space $\mathscr{T}_2^0 M$,

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{C.5}$$

where now the dx^μ is not the difference between coordinates of p and q which most physicists will imagine it as the infinitesimal distance, but it should be considered as the basis of dual vectors in M. Being a $(0,2)$-tensor, it needs two vectors to produce a real number, and if we provide the vector ds for its argument we will get the equation (C.4) above. Since dx^μ and dx^ν can be interchanged, then we need the metric g to be symmetric in its indices,

$$g_{\mu\nu} = g_{\nu\mu} \tag{C.6}$$

If we have two vectors[1] $U, V \in T_pM$, then the inner product between

[1]Note that these two vectors need to be in the same neighborhood U of a certain point $p \in M$ to ensure that we use the same coordinate system for both of them.

them is

$$\begin{aligned} g(U,V) &= g_{\mu\nu}U^\alpha V^\beta (dx^\mu \otimes dx^\nu)(e_\alpha, e_\beta) \\ &= g_{\mu\nu}U^\alpha V^\beta \delta^\mu_\alpha \delta^\nu_\beta \\ &= g_{\mu\nu}U^\mu V^\nu \end{aligned} \tag{C.7}$$

such that if we set $V = U$, then we have the length of vector U,

$$|U|^2 = g(U,U) = g_{\mu\nu}U^\mu U^\nu \tag{C.8}$$

But it still doesn't tell us the whole story. We can restrict the metric g to be positive-definite, in the sense that if we have a vector $U \in T_pM$, then the norm of this vector $|U|^2$ must be nonnegative, or $|U|^2 = g(U,U) \geq 0$, where the equation holds only for the case of the nullity of U. If a manifold M is equipped with the metric g which is symmetric and positive-definite, then this manifold is called the *Riemannian manifold*, and its metric g is called the *Riemannian metric*. Since it is positive-definite, all eigenvalues of Riemannian metric are positive, and hence it has the inverse denoted as $g^{\mu\nu}$. Moreover, it also can be diagonalized by using certain procedure involving the orthogonal matrix, and rescaled such that all entries in the diagonal of diagonalized metric are the unity. This matrix is nothing other than the identity metric, which is also known as the metric tensor of the Euclidean space $\mathbb{R}^n$, denoted as $\delta_{\mu\nu}$. Therefore, we conclude that the Riemannian metric g of a Riemannian manifold M can be reduced to the identity metric δ of the Euclidean space $\mathbb{R}^n$.

We can weaken our restriction about the positive-definiteness of metric to a more flexible condition: $g(U,U)$ can be zero although U is not zero, and it is possible to have $g(U,U) < 0$ for a certain $U \in T_pM$. If a manifold M is equipped with this kind of symmetric metric, then M is called *pseudo-Riemannian manifold*, and its metric g is called *pseudo-Riemannian metric*. We can principally reduce this metric into the diagonal matrix, and hence we will get some positive and negative eigenvalues, or after rescaling and reordering, we will have the diagonal matrix $\eta = diag(1, \ldots, 1, -1, \ldots, -1)$, in which there are r numbers of 1 and s numbers of -1. The *signature* of metric is defined as $r - s$, and it is same for all points in the manifold. For the specific case, if the number of positive eigenvalues is only one, then M is called the *Lorentzian manifold*, and its metric g is called the *Lorentzian metric*. The diagonal metric corresponding to the Lorentzian one is called the *Minkowski metric*, $\eta = diag(1, -1, \ldots, -1)$.

Let's back to the case of Riemannian manifold M with a Riemannian metric g. If this metric is provided only with one vector $U = U^\mu \partial/\partial x^\mu \in T_pM$, i.e. $g(U, \cdot)$, then it will produce something like the dual vector, since this object needs one more vector to produce a real number. Therefore, the metric g can be a bridge between the set of vectors and dual vectors in the manifold. If we denote the dual vector corresponding to $g(U, \cdot)$ as $U_\nu dx^\nu$, then

$$U_\nu dx^\nu = g(U, \cdot) \tag{C.9}$$

such that componentwise it has the form

$$U_\nu = g_{\mu\nu} U^\mu \tag{C.10}$$

If the vector $V \in T_pM$ is taken into the empty slot of $g(U, \cdot)$, then we have

$$g_{\mu\nu} U^\mu V^\nu = U_\nu V^\nu \tag{C.11}$$

where the operation of inner product now can be carried out easily by using this new tool.

Here is the clear ground for this matter. In the manifold where we don't introduce the metric, the inner product takes place between a vector and a dual vector. And since there is no link between the set of tangential vectors T_pM and dual vectors T_p^*M, we can take a product between any vectors and any dual vectors without an interesting intuitive sense. If we are working in the (pseudo-)Riemannian manifold, then the inner product also takes place between a vector and a dual vector. But since there is a linking chain between them, so called as the metric, then we will have an interesting situation when we take an inner product between a vector and its corresponding dual, in which it will give us the norm or length of that vector (or, dual vector). Of course we can freely take an inner product between a vector with any dual vectors which don't correspond to that vector, in which we will have the *ordinary* inner product between two different vectors.

C.2 The Concepts of Connection, Covariant Derivative and Parallel Transport

Now suppose we have a Riemannian manifold M equipped with a Riemannian metric g, with a neighborhood $U \subset M$ which contains $p \in M$ such that the coordinate system in this neighborhood is $\{x^\mu\}$. If in this manifold there exists a vector field $V \in \chi M$, then the component of this vector in the neighborhood U is $V = V^\mu \partial/\partial x^\mu = V^\mu e_\mu$. What if we differentiate this vector in a direction of x^ν? It has no problem when the manifold is flat, but since we are working on the general case, we must remember that the coordinate system in another neigborhood adjacent to U is different with U. Then, what we can do at best is

$$\partial_\nu V = \partial_\nu (V^\mu e_\mu) = (\partial_\nu V^\mu) e_\mu + V^\mu (\partial_\nu e_\mu) \tag{C.12}$$

The first term on the RHS of equation above is what we expect to get if the manifold is flat. But the second term is rather strange. Indeed, we don't omit it from the equation because it doesn't vanish. The nonvanishing property of term $\partial_\nu e_\mu$ tells us that the basis vectors are not constant throughout the manifold. Intuitively the gradient of basis vectors doesn't lie intrinsically in the manifold; if we embbed the manifold in some other familiar spaces like a Euclidean, then there will be a component of this gradient which is orthogonal out of the manifold. We don't talk about the complete vector of gradient of basis vectors, but we only work with its parallel component which lies along the manifold, since the direction out of manifold is not its intrinsic property because it really depends on how we embbed the manifold. Since the component parallel with the local manifold is also a vector, then we can describe it as the linear superposition of the basis vectors,

$$\partial_\nu e_\mu = \Gamma^\lambda{}_{\nu\mu} e_\lambda \tag{C.13}$$

such that now we will get

$$\nabla_\nu V \equiv \partial_\nu V = (\partial_\nu V^\mu + V^\rho \Gamma^\mu{}_{\nu\rho}) e_\mu \tag{C.14}$$

where we will call it as the *covariant derivative*, which reminds us that it is the proper derivative in the manifold, hence it will be provided the new symbol ∇ to differ with our ordinary differentiation in the flat space.

Since it is the differentiation along the direction of x^ν, which means the direction of e_ν, the symbol $\nabla_\nu V$ can also be written as $\nabla_{e_\nu} V$. It really helps us to formulate the differentiation of a vector along the direction of a vector field, say $X = X^\mu \partial/\partial x^\mu$, instead of only one direction of the basis vector. Then obviously for this general case we will have

$$\nabla_X V = \nabla_{(X^\nu e_\nu)} V = X^\nu \nabla_{e_\nu} V \tag{C.15}$$

and by inserting the previous result for $\nabla_{e_\nu} V$, it yields

$$\nabla_X V = X^\nu (\partial_\nu V^\mu + V^\rho \Gamma^\mu{}_{\nu\rho}) e_\mu \tag{C.16}$$

The operator ∇ above is called the *affine connection*, and it is indeed a mapping $\nabla : \chi M \times \chi M \to \chi M$, which has the properties

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \tag{C.17}$$

$$\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z \tag{C.18}$$

$$\nabla_{(fX)} Y = f \nabla_X Y \tag{C.19}$$

$$\nabla_X (fY) = X[f] Y + f \nabla_X Y \tag{C.20}$$

for a function $f \in \mathcal{F}M$ and $X, Y, Z \in TM$. These properties can be checked directly from the defining equation (C.16).

We can view the equation (C.16) just like what ordinary differentiation means; it tells us about the change of the vector field V when we move along the integral curve of the vector field X. Notice that it doesn't mean to bring or translate a single vector V along that curve, but we do compare the two different vectors in different points in the curve where those two vectors belong to the vector field V. If the covariant derivative of V along the integral curve of X is zero, then we call that the vector field V is *parallel transported* along the integral curve of X, where the name is suggested from the fact that the vectors of V in two different points along the curve is parallel. The equation which defines this parallel transport is

$$\frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}V^\rho = 0 \tag{C.21}$$

where we have used the tangentiality condition for X, i.e. $X = d/dt = (dx^\nu/dt)e_\nu$. Moreover, if we set $V = X$ in equation (C.21) above, we will get

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \tag{C.22}$$

The curve $x^\mu(t)$ which satisfies the equation (C.22) is called *geodesic*. It tells us that in geodesic, the tangential vectors in every points along this curve point to the same direction, hence it will be the candidate for the straightest curve in the curved manifold. By using the calculus of variation, we can also show that the curve which has the shortest length between any two points in the manifold is this geodesic defined by equation (C.22), and therefore we can state safely that the generalization of straight line in the flat space to the curved manifold is the geodesic, where both of them share the same properties: they are the straightest and the shortest paths which connect any two points in the space. We also can show easily that the equation (C.22) above will be reduced to the one which characterizes the straight line in the flat space,

$$\frac{d^2x^\mu}{dt^2} = 0 \tag{C.23}$$

where we have noted that $\Gamma^\mu{}_{\nu\rho}$, being the component of the gradient of basis vectors in the manifold, should be zero in the flat space.

The covariant derivative can be applied to a case of function on the manifold. If we have $f \in \mathcal{F}M$, then its covariant derivative along the integral curve of the vector field X is

$$\nabla_X f = X[f] \tag{C.24}$$

such that if we have a vector field V, the covariant derivative of fV is

$$\nabla_X(fV) = X[f]V + f\nabla_X V \tag{C.25}$$

where it is similar with the Leibniz rule for the differentiation of the product functions. Then, if we set $f = \langle \omega, V \rangle$ for a dual vector $\omega \in T_p^*M$, we have

$$\nabla_X \langle \omega, V \rangle = X[\langle \omega, V \rangle] = \langle \nabla_X \omega, V \rangle + \langle \omega, \nabla_X V \rangle \tag{C.26}$$

If we write the components of the equations above, we will have

$$X^\mu((\partial_\mu \omega_\nu)V^\nu + \omega_\nu(\partial_\mu V^\mu)) = (\nabla_X \omega)_\mu V^\mu + \omega_\mu X^\nu \partial_\nu V^\mu + \omega_\mu X^\nu V^\rho \Gamma^\mu{}_{\nu\rho} \tag{C.27}$$

and it yields

$$(\nabla_X \omega)_\mu = X^\nu(\partial_\nu \omega_\mu - \omega_\rho \Gamma^\rho{}_{\nu\mu}) \tag{C.28}$$

Therefore, we get the complete description for the covariant derivative of dual vector ω,

$$\nabla_X \omega = X^\nu(\partial_\nu \omega_\mu - \omega_\rho \Gamma^\rho{}_{\nu\mu})dx^\mu \tag{C.29}$$

Notice that the appearance of the second term in the RHS of above equation is due to the nonvanishing property of the gradient of basis of dual vectors in M. Namely, since

$$\nabla_{e_\nu} \omega = \nabla_\nu \omega = \partial_\nu \omega_\mu dx^\mu + \omega_\mu \partial_\nu dx^\mu \tag{C.30}$$

where dx^μ should be considered as the basis of dual vectors, not the infinitesimal length or the difference between two values of coordinates. From this point we can see that $\partial_\nu dx^\mu$ must satisfy

$$\partial_\nu dx^\mu = -\Gamma^\mu{}_{\nu\lambda} dx^\lambda \tag{C.31}$$

which is analog with the case of basis of vectors.

For any tensors $T_1, T_2 \in \mathscr{T}M$, we require that

$$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2) \tag{C.32}$$

and the generalization for the case of tensor is rather straighforward. As an example, the covariant derivative of metric g is

$$(\nabla_\lambda g)_{\mu\nu} = \partial_\lambda g_{\mu\nu} - g_{\rho\nu}\Gamma^\rho{}_{\lambda\mu} - g_{\mu\rho}\Gamma^\rho{}_{\lambda\nu} \tag{C.33}$$

C.3 Connection Coefficient, Metric and Levi-Civita Connections

We will call $\Gamma^{\mu}{}_{\nu\rho}$ as the *connection coefficient*. It is an object that has indices, but actually it is not a tensor. Recall that if an object T of the same type with this connection coefficient has a component $T^{\mu}{}_{\nu\rho}$ in the coordinate system $\{x^{\mu}\}$ and $T'^{\mu}{}_{\nu\rho}$ in $\{x'^{\mu}\}$, then

$$T'^{\mu}{}_{\nu\rho} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\omega}}{\partial x'^{\rho}}T^{\alpha}{}_{\beta\omega} \tag{C.34}$$

then T is a tensor. Therefore, we need to check the transformation property of the connection coeeficient if we change the coordinate system.

In the coordinate system $\{x'^{\mu}\}$, the gradient of basis vectors $f_{\mu} \equiv \partial/\partial x'^{\mu}$ can be stated as

$$\frac{\partial}{\partial x'^{\nu}}f_{\mu} = \Gamma'^{\lambda}{}_{\nu\mu}f_{\lambda} \tag{C.35}$$

where $\Gamma'^{\lambda}{}_{\nu\mu}$ is the connection coefficient in this prime coordinate. Since

$$\frac{\partial}{\partial x'^{\nu}}f_{\mu} = \frac{\partial}{\partial x'^{\nu}}\Big(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}e_{\alpha}\Big) = \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}x'^{\mu}}e_{\alpha} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial}{\partial x^{\beta}}e_{\alpha} \tag{C.36}$$

then it implies

$$\begin{aligned}
\Gamma'^{\lambda}{}_{\nu\mu}f_{\lambda} &= \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}x'^{\mu}}e_{\alpha} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial}{\partial x^{\beta}}e_{\alpha} \\
\Gamma'^{\lambda}{}_{\nu\mu}\frac{\partial x^{\alpha}}{\partial x'^{\lambda}}e_{\alpha} &= \Big(\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}x'^{\mu}} + \frac{\partial x^{\omega}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\Gamma^{\alpha}{}_{\beta\omega}\Big)e_{\alpha} \\
\Gamma'^{\lambda}{}_{\nu\mu} &= \frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}x'^{\mu}} + \frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\omega}}{\partial x'^{\mu}}\Gamma^{\alpha}{}_{\beta\omega}
\end{aligned} \tag{C.37}$$

The appearance of the first term in the RHS of equation above makes the connection coefficient failed to be a tensor.

Now we want to put a restriction for our form of connection coefficient. If we parallel transport the two vectors V and W along the integral curve of X, where $V, W, X \in TM$, then we require that the inner product between V and W doesn't change along the transportation. Precisely,

$$\begin{aligned}
0 &= \nabla_X(g(V,W)) = (\nabla_X g)(V,W) + g(\nabla_X V, W) + g(V, \nabla_X W) \\
&= X^{\lambda}(\nabla_{\lambda} g)_{\mu\nu}V^{\mu}W^{\nu}
\end{aligned} \tag{C.38}$$

where we have used the fact that our vectors V and W are parallel transported, $\nabla_X V = \nabla_X W = 0$. Therefore, the covariant derivative of metric g is zero,

$$(\nabla_\lambda g)_{\mu\nu} = 0 \tag{C.39}$$

If the connection ∇ satisfies the equation (C.39) above, then it is called the *metric connection*. From equation (C.33) we can write

$$\partial_\lambda g_{\mu\nu} - g_{\rho\nu}\Gamma^\rho{}_{\lambda\mu} - g_{\mu\rho}\Gamma^\rho{}_{\lambda\nu} = 0 \tag{C.40}$$

By cycling the indices, we have

$$\begin{aligned}
\partial_\mu g_{\nu\lambda} - g_{\rho\lambda}\Gamma^\rho{}_{\mu\nu} - g_{\nu\rho}\Gamma^\rho{}_{\mu\lambda} &= 0 &\text{(C.41)}\\
\partial_\nu g_{\lambda\mu} - g_{\rho\mu}\Gamma^\rho{}_{\nu\lambda} - g_{\lambda\rho}\Gamma^\rho{}_{\nu\mu} &= 0 &\text{(C.42)}
\end{aligned}$$

such that $-$(C.40) + (C.41) + (C.42) yields

$$-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} + g_{\rho\nu}T^\rho{}_{\lambda\mu} + g_{\rho\mu}T^\rho{}_{\lambda\nu} - 2g_{\rho\lambda}\Gamma^\lambda{}_{(\mu\nu)} = 0 \tag{C.43}$$

where we have defined

$$\begin{aligned}
T^\rho{}_{\lambda\mu} &\equiv \Gamma^\rho{}_{\lambda\mu} - \Gamma^\rho{}_{\mu\lambda} &\text{(C.44)}\\
\Gamma^\rho{}_{(\mu\nu)} &\equiv \frac{1}{2}\Big(\Gamma^\rho{}_{\mu\nu} + \Gamma^\rho{}_{\nu\mu}\Big) &\text{(C.45)}
\end{aligned}$$

The quantity $T^\rho{}_{\lambda\mu}$ is called the *torsion tensor*, and it is indeed a tensor, which can be proved easily by using the transformation property of connection coefficient (C.37).

The description for $\Gamma^\rho{}_{(\mu\nu)}$ can be obtained easily,

$$\Gamma^\rho{}_{(\mu\nu)} = \begin{Bmatrix} \rho \\ \mu\nu \end{Bmatrix} + \frac{1}{2}(T_\nu{}^\rho{}_\mu + T_\mu{}^\rho{}_\nu) \tag{C.46}$$

where we have defined the *Christoffel symbol* as

$$\begin{Bmatrix} \rho \\ \mu\nu \end{Bmatrix} \equiv \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{C.47}$$

Now we can calculate the formula for the connection coefficient

$$\begin{aligned}
\Gamma^\rho{}_{\mu\nu} &= \Gamma^\rho{}_{(\mu\nu)} + \frac{1}{2}T^\rho{}_{\mu\nu} &\text{(C.48)}\\
&= \begin{Bmatrix} \rho \\ \mu\nu \end{Bmatrix} + K^\rho{}_{\mu\nu} &\text{(C.49)}
\end{aligned}$$

The quantity $K^{\rho}{}_{\mu\nu}$ is called the *contorsion*, defined as

$$K^{\rho}{}_{\mu\nu} \equiv \frac{1}{2}(T^{\rho}{}_{\mu\nu} + T_{\nu}{}^{\rho}{}_{\mu} + T_{\mu}{}^{\rho}{}_{\nu}) \tag{C.50}$$

It is obviously a tensor.

There is an interesting fact about this matter. If we have a Riemannian metric M equipped with a Riemannian metric g with its covariant derivative is zero and the torsion tensor T vanishes, i.e. $\Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu}$, then the connection ∇ is called the *Levi-Civita connection.* A well-known theorem states that for any Riemannian manifold in which these conditions hold except the vanishing of torsion tensor, then there exists uniquely a Levi-Civita connection. It comes from the fact that if we are given the connection coefficient $\Gamma^{\rho}{}_{\mu\nu}$, then the quantity

$$\Gamma'^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + S^{\rho}{}_{\mu\nu} \tag{C.51}$$

is also a connection coefficient if S is tensor. Then by choosing $S^{\rho}{}_{\mu\nu}$ to be

$$S^{\rho}{}_{\mu\nu} = -K^{\rho}{}_{\mu\nu} \tag{C.52}$$

we can see that from equation (C.49) the connection coefficient $\Gamma'^{\rho}{}_{\mu\nu}$ now becomes

$$\Gamma'^{\rho}{}_{\mu\nu} = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} \tag{C.53}$$

where it is obviously the connection coefficient of the Levi-Civita connection.

C.4 Torsion and Curvature Tensors

We have talked so far about the properties which differ the curved manifold with the flat one, such as whether the connection coefficient vanishes or not, but we still don't have the intrinsic mechanicsm which allows us to determine the curvature of manifold, in the sense that this mechanics must not rely on the coordinate system heavily. The nonvanishing connection coefficient cannot provide these requirements for us for two reasons. First, because it depends on how we make the system of coordinate basis vectors over the manifold and it ends up with how we choose the coordinate system. And second, because it cannot tell us how much the curvature of some region of a manifold. In this subsection we will discuss about the properties of torsion tensor and some curvature tensors. We also discuss the torsion tensor here because its geometrical meaning is as important as the curvature tensors. Here, we will talk about torsion tensor first before attacking the curvature tensors because of the former is simpler than the latter.

The torsion tensor $T : \chi M \times \chi M \rightarrow \chi M$ is defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{C.54}$$

for the vector fields $X, Y \in \chi M$. Componentwise, it is described as

$$\begin{aligned} T^\lambda{}_{\mu\nu} e_\lambda &= T(e_\mu, e_\nu) = \nabla_\mu e_\nu - \nabla_\nu e_\mu - [e_\mu, e_\nu] \\ &= (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}) e_\lambda \end{aligned} \tag{C.55}$$

such that we have

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \tag{C.56}$$

Therefore, torsion tensor measures how much we differ from the symmetric property of connection coefficient in the Levi-Civita case. If $T^\lambda{}_{\mu\nu} = 0$, then the manifold is called *torsionless manifold*.

If we have three points $p, q, r \in M$ with their coordinates $\{x^\mu\}$, $\{x^\mu + \delta^\mu\}$ and $\{x^\mu + \epsilon^\mu\}$ respectively, with $\delta^\mu, \epsilon^\mu \ll$ for all μ. If we construct the vectors $X = \delta^\mu e_\mu$ and $Y = \epsilon^\mu e_\mu$ such that X (Y) is the vector which connects the point p to q (r). If we parallel transport the vector X (Y) along the infinitesimal line pr (pq), then the component of vector X (Y) becomes $\delta^\mu - \delta^\lambda \epsilon^\nu \Gamma^\mu{}_{\nu\lambda}$ $(\epsilon^\mu - \epsilon^\nu \delta^\lambda \Gamma^\mu{}_{\lambda\nu})$. This transported vector of X (Y) connects a point r (q) to a new point s_1 (s_2), such that we have the vector which connects points p and s_1 (p and s_2) is $pr + rs_1$ $(pq + qs_2)$, i.e.

$$pr + rs_1 = \epsilon^\mu + \delta^\mu - \delta^\lambda \epsilon^\nu \Gamma^\mu{}_{\nu\lambda} \tag{C.57}$$

$$pq + qs_2 = \delta^\mu + \epsilon^\mu - \epsilon^\nu \delta^\lambda \Gamma^\mu{}_{\lambda\nu} \tag{C.58}$$

The difference between the vectors ps_1 and ps_2 is

$$\delta^\lambda \epsilon^\nu (\Gamma^\mu{}_{\lambda\nu} - \Gamma^\mu{}_{\nu\lambda}) \tag{C.59}$$

where we notice that the term inside the parenthesis is the torsion tensor $T^{\mu}{}_{\lambda\nu}$. Hence, if the torsion tensor vanishes, then the parallelogram made of small displacement vectors and their parallel transport is closed.

The first curvature tensor I want to discuss is the *Riemann curvature tensor* $R : \chi M \times \chi M \times \chi M \to \chi M$. It is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad \text{(C.60)}$$

for $X, Y, Z \in \chi M$. The component of Riemann tensor is described as

$$R^{\lambda}{}_{\mu\nu\sigma} e_{\lambda} = \nabla_{\nu} \nabla_{\sigma} e_{\mu} - \nabla_{\sigma} \nabla_{\nu} e_{\mu} - \nabla_{[e_\nu, e_\sigma]} e_{\mu} \qquad \text{(C.61)}$$

$$= (\partial_{\nu} \Gamma^{\lambda}{}_{\sigma\mu} - \partial_{\sigma} \Gamma^{\lambda}{}_{\nu\mu} + \Gamma^{\omega}{}_{\sigma\mu} \Gamma^{\lambda}{}_{\nu\omega} - \Gamma^{\omega}{}_{\nu\mu} \Gamma^{\lambda}{}_{\sigma\omega}) e_{\lambda} \qquad \text{(C.62)}$$

such that we have

$$R^{\lambda}{}_{\mu\nu\sigma} = \partial_{\nu} \Gamma^{\lambda}{}_{\sigma\mu} - \partial_{\sigma} \Gamma^{\lambda}{}_{\nu\mu} + \Gamma^{\omega}{}_{\sigma\mu} \Gamma^{\lambda}{}_{\nu\omega} - \Gamma^{\omega}{}_{\nu\mu} \Gamma^{\lambda}{}_{\sigma\omega} \qquad \text{(C.63)}$$

The geometrical meaning of Riemann tensor is about the difference of two vectors after they are parallel transported from the same vector along two different paths. Suppose we have four points p, q, r, s which have coordinates $\{x^{\mu}\}$, $\{x^{\mu} + \delta^{\mu}\}$, $\{x^{\mu} + \epsilon^{\mu}\}$ and $\{x^{\mu} + \delta^{\mu} + \epsilon^{\mu}\}$ respectively. If we have a vector $V \in T_p M$, then after parallel transport to point s along the path pqs, this vector has the form

$$V^{\mu} - V^{\kappa} \delta^{\nu} \Gamma^{\mu}{}_{\nu\kappa} - V^{\kappa} \epsilon^{\nu} \Gamma^{\mu}{}_{\nu\kappa} - V^{\kappa} \epsilon^{\lambda} \delta^{\nu} (\partial_{\nu} \Gamma^{\mu}{}_{\lambda\kappa} - \Gamma^{\omega}{}_{\nu\kappa} \Gamma^{\mu}{}_{\lambda\omega}) \qquad \text{(C.64)}$$

and similarly for the path prs,

$$V^{\mu} - V^{\kappa} \epsilon^{\nu} \Gamma^{\mu}{}_{\nu\kappa} - V^{\kappa} \delta^{\nu} \Gamma^{\mu}{}_{\nu\kappa} - V^{\kappa} \epsilon^{\lambda} \delta^{\nu} (\partial_{\lambda} \Gamma^{\mu}{}_{\nu\kappa} - \Gamma^{\omega}{}_{\lambda\kappa} \Gamma^{\mu}{}_{\nu\omega}) \qquad \text{(C.65)}$$

where we omit the terms larger than second order in δ and ϵ. The difference between these two vectors is

$$V^{\kappa} \epsilon^{\lambda} \delta^{\nu} (\partial_{\lambda} \Gamma^{\mu}{}_{\nu\kappa} - \partial_{\nu} \Gamma^{\mu}{}_{\lambda\kappa} + \Gamma^{\omega}{}_{\nu\kappa} \Gamma^{\mu}{}_{\lambda\omega} - \Gamma^{\omega}{}_{\lambda\kappa} \Gamma^{\mu}{}_{\lambda\omega}) \qquad \text{(C.66)}$$

and again, we can identify the term in parenthesis as the Riemann tensor $R^{\mu}{}_{\kappa\lambda\nu}$.

The *Ricci curvature tensor* is defined as

$$Ric(X, Y) \equiv \langle dx^{\mu}, R(e_{\mu}, Y)X\rangle \tag{C.67}$$

and its component is

$$R_{\mu\nu} = Ric(e_{\mu}, e_{\nu}) = R^{\lambda}{}_{\mu\lambda\nu} \tag{C.68}$$

such that we can form the *Ricci scalar curvature*, defined as the contraction of Ricci curvature tensor with the metric,

$$R = g^{\mu\nu} R_{\mu\nu} \tag{C.69}$$