UNIVERSITAS INDONESIA

PENAKSIRAN *MEAN SQUARED ERROR* (MSE) *EMPIRICAL BEST LINIER UNBIASED PREDICTION* (EBLUP) PADA MODEL FAY-HERRIOT

SKRIPSI

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FAKULTAS MATEMATIKA DAN ILMU PENGETAHUAN ALAM
PROGRAM STUDI SARJANA MATEMATIKA
DEPOK
JUNI 2012

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Diajukan sebagai salah satu syarat untuk memperoleh gelar sarjana sains

LUTHFATUL AMALIANA
0806452210

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Tanggal	:	19 Juni 2012

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NPM	:	0806452210
Program Studi	:	Sarjana Matematika
Judul Skripsi	:	Penaksiran *Mean Squared Error* (MSE) *Empirical Best Linear Unbiased Prediction* (EBLUP) pada Model Fay-Herriot

Telah berhasil dipertahankan di hadapan Dewan Penguji dan diterima sebagai bagian persyaratan yang diperlukan untuk memperoleh gelar Sarjana Sains pada Program Studi S1 Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam Universitas Indonesia

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ABSTRAK

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Judul : Penaksiran *Mean Squared Error* (MSE) *Empirical Best Linear Unbiased Predictor* (EBLUP) pada Model Fay-Herriot

Model Fay-Herriot merupakan salah satu kasus khusus dari model area level dalam *small area estimation* (SAE). Penaksiran parameter pada model Fay-Herriot dapat dilakukan dengan beberapa metode, diantaranya metode BLUP dan EBLUP. Metode yang digunakan pada tugas akhir ini adalah metode EBLUP, sehingga dapat diperoleh penaksir EBLUP. Untuk mengukur seberapa baik taksiran EBLUP yang diperoleh, akan dicari nilai MSE EBLUP. Namun, nilai MSE EBLUP tersebut masih bergantung pada variansi pengaruh acak *small area* yang tidak diketahui. Oleh karena itu, dilakukan penaksiran terhadap MSE EBLUP dengan mensubstitusikan taksiran variansi pengaruh acak *small area* ke dalam MSE EBLUP, sehingga diperoleh taksiran MSE EBLUP.

Kata Kunci : SAE, model Fay-Herriot, penaksir EBLUP, MSE EBLUP, taksiran MSE EBLUP.
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ABSTRACT

Name : Luthfatul Amaliana
Program Study : Mathematics
Title : The Estimation of Mean Squared Error (MSE) Empirical Best Linear Unbiased Prediction (EBLUP) in Fay-Herriot Model

Fay-Herriot model is one of the special case of basic area level model in small area estimation (SAE). Parameter in Fay-Herriot model can be estimated with many methods, such as BLUP and EBLUP method. In this minithesis, the method will be used is EBLUP method and then the EBLUP estimator can be gotten. The MSE EBLUP will be found to measure how good the EBLUP estimator is. However, it still depends on variance of random effect of small area which is unknown. Therefore, the MSE EBLUP will be estimated by substituting the estimated variance of random effect of small area to the MSE EBLUP, so that the estimator of MSE EBLUP is obtained.

Keywords : SAE, Fay-Herriot model, EBLUP estimator, MSE EBLUP, MSE EBLUP estimator.
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DAFTAR LAMPIRAN

BAB 1
PENDAHULUAN

1.1 Latar Belakang

Small area didefinisikan sebagai suatu subpopulasi kecil atau gambaran suatu area geografis kecil (Rao, 2003). *Small area* dapat berupa desa/kelurahan, kecamatan, kabupaten, kelompok suku, maupun kelompok umur. Suatu teknik statistika yang memanfaatkan informasi tambahan (*auxiliary variables*), seperti data sensus dan atau catatan administratif *small area* tersebut, maupun catatan administratif *small area* lain yang memiliki karakteristik hampir sama dinamakan *small area estimation* (Rao, 2003).

Penaksiran parameter pada *small area* dapat dilakukan dengan menggunakan model desain penarikan sampel, yang disebut sebagai penaksiran langsung (*direct estimation*), dan penaksiran tidak langsung (*indirect estimation*). Namun, penaksiran langsung tidak dapat menghasilkan ketelitian yang cukup apabila ukuran sampel dari *small area* yang diamati kecil, sehingga untuk memperkecil *standar error* penaksiran, diperlukan metode penaksiran tidak langsung (*indirect estimation*).

Pada penaksiran tidak langsung, terdapat dua jenis model penghubung yaitu model implisit dan eksplisit. Model implisit menghubungkan *small area* yang satu dengan *small area* lainnya melalui informasi tambahan (*auxiliary variables*). Metode yang dapat digunakan pada model implisit diantaranya metode sintetik dan komposit, yang masing-masing akan menghasilkan penaksir sintetik dan komposit.

Berbeda dengan model implisit, model eksplisit termasuk dalam *mixed model* yang mengandung pengaruh acak *small area*. Pengaruh acak tersebut menjelaskan variasi antar *small area* yang belum dijelaskan oleh *auxiliary variables* yang terkandung di dalam model. Selanjutnya model eksplisit ini disebut dengan model *small area*. Berdasarkan ketersediaan datanya, model *small area* terbagi menjadi model area level dan model unit level. Salah satu kasus khusus dari model area level adalah model Fay-Herriot. Dalam tugas akhir ini, model yang akan digunakan adalah model Fay-Herriot.

Metode penaksiran yang dapat digunakan pada model Fay-Herriot yaitu metode Bayes dan non-Bayes, dimana metode Bayes terdiri dari *Empirical Bayes* (EB) dan *Hierarchical Bayes* (HB). Sedangkan metode non-Bayes antara lain metode BLUP (*Best Linear Unbiased Prediction*) dan EBLUP (*Empirical Best Linear Unbiased Prediction*). Pada metode BLUP, variansi pengaruh acak *small area* diasumsikan telah diketahui, sedangkan pada metode EBLUP tidak diketahui nilainya dan harus ditaksir. Metode EBLUP inilah yang akan digunakan dalam penaksiran pada model Fay-Herriot di tugas akhir ini.

Salah satu ukuran untuk mengetahui seberapa baik taksiran parameter yang diperoleh dalam suatu penaksiran adalah dengan melihat nilai *Mean Squared Error* (MSE) dari taksiran yang diperoleh. Penaksiran parameter pada model Fay-Herriot dengan metode EBLUP akan menghasilkan MSE EBLUP. Namun, nilai MSE EBLUP ini masih bergantung pada variansi pengaruh acak yang tidak diketahui nilainya, sehingga perlu dilakukan penaksiran pada MSE EBLUP yang telah diperoleh. Oleh karena itu, pada tugas akhir ini akan dibahas mengenai penaksiran MSE EBLUP pada model Fay-Herriot dengan metode EBLUP.

1.2 Perumusan Masalah

Dalam tugas akhir ini, akan diselesaikan masalah bagaimana mencari taksiran MSE pada model Fay-Herriot dengan metode EBLUP.

1.3 Tujuan Penelitian

Tujuan dari penulisan tugas akhir ini adalah mencari taksiran MSE pada model Fay-Herriot dengan metode EBLUP.

1.4 Pembatasan Masalah

Permasalahan dalam tugas akhir ini dibatasi pada variabel acak yang akan digunakan yaitu variabel acak kontinu dan penaksiran yang akan dilakukan didasarkan pada model (*model-based*), bukan didasarkan pada sampel (*sample-based*).

BAB 2
LANDASAN TEORI

Pada bab ini, akan dibahas mengenai teori-teori dasar yang mendukung dan akan digunakan untuk mencari taksiran *mean squared error* (MSE) pada model Fay-Herriot. Teori-teori tersebut antara lain mengenai definisi MSE, metode *lagrange*, bentuk kuadratik dan definit positif, matriks partisi, differensiasi matriks, definisi *translation-invariant* dan *maximal-invariant*, *general linear mixed model* (GLMM), metode BLUP dan MSE BLUP pada GLMM, serta metode EBLUP dan taksiran MSE EBLUP pada GLMM.

2.1 Definisi *Mean Squared Error* (MSE)

Misalkan θ merupakan suatu parameter dan $\hat{\theta}$ merupakan taksiran dari parameter θ. MSE dari $\hat{\theta}$ didefinisikan sebagai berikut:

Misalkan: $\mathrm{E}\big[\hat{\theta}\big] = \mathrm{a}$, yang belum tentu θ,

$$\begin{aligned}
\mathrm{MSE}[\hat{\theta}] &= \mathrm{E}[(\hat{\theta}-\theta)^2] \\
&= \mathrm{E}[(\hat{\theta}-\mathrm{a}+\mathrm{a}-\theta)^2] \\
&= \mathrm{E}[\big(\hat{\theta}-\mathrm{a}\big)^2 + 2\big(\hat{\theta}-\mathrm{a}\big)(\mathrm{a}-\theta) + (\mathrm{a}-\theta)^2] \\
&= \mathrm{E}[\big(\hat{\theta}-\mathrm{a}\big)^2] + 2\mathrm{E}[\big(\hat{\theta}-\mathrm{a}\big)](\mathrm{a}-\theta) + \mathrm{E}[(\mathrm{a}-\theta)^2] \\
&= \mathrm{E}[\big(\hat{\theta}-\mathrm{a}\big)^2] + 2\,\underbrace{\mathrm{E}[\big(\hat{\theta}-\mathrm{a}\big)]}_{=0}\,(\mathrm{a}-\theta) + \mathrm{E}(\mathrm{a}-\theta)^2
\end{aligned}$$

$$\mathrm{MSE}[\hat{\theta}] = \mathrm{var}[\hat{\theta}] + [\mathrm{bias}(\hat{\theta})^2] \quad ;\text{karena}2\mathrm{E}\big[\big(\hat{\theta}-\mathrm{a}\big)\big] = 0 \qquad (2.1)$$

Berdasarkan definisi MSE tersebut, jika $\hat{\theta}$ yang diperoleh *unbiased*, maka MSE dari $\hat{\theta}$ akan sama dengan variansi dari $\hat{\theta}$. Sedangkan *standar error* dari $\hat{\theta}$ didefinisikan sebagai akar kuadrat positif dari $\mathrm{MSE}[\hat{\theta}]$.

Nilai MSE dari suatu taksiran parameter memiliki peranan penting untuk diketahui, diantaranya adalah untuk mengukur seberapa baik taksiran parameter yang diperoleh (Rao, 2003).

2.2 Metode *Lagrange*

Metode *Lagrange* merupakan suatu metode yang digunakan untuk memaksimumkan atau meminimumkan suatu fungsi $f(\mathbf{x})$ terhadap kendala $g(\mathbf{x})$ dengan cara menyelesaikan sistem persamaan berikut:

$$\nabla f(\mathbf{x}) = \boldsymbol{\eta}\nabla g(\mathbf{x}) \text{ dan } g(\mathbf{x}) = 0$$

sehingga didapat $\mathbf{x}$ dan $\boldsymbol{\eta}$ yang meminimumkan atau memaksimumkan fungsi $f(\mathbf{x})$. Setiap $\mathbf{x}$ yang memenuhi sistem persamaan di atas disebut vektor kritis untuk masalah nilai ekstrim dengan kendala. Sedangkan $\boldsymbol{\eta}$ disebut pengali *Lagrange* dan $\nabla f(\mathbf{x}) = \frac{\partial f}{\partial x_1}\mathbf{i}_1 + \frac{\partial f}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{i}_n$ (Purcell, 2000).

2.3 Bentuk Kuadratik dan Definit Positif

Misalkan matriks $\mathbf{A} = [a_{ij}]$ berukuran nxn. Pandang fungsi berikut:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i,j} a_{ij}\mathrm{x}_i\mathrm{x}_j = \sum_i a_i\mathrm{x}_i^2 + \sum_{i,j\neq i} a_{ij}\mathrm{x}_i\mathrm{x}_j$$

dengan $\mathbf{x}^T = (\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_n)$ di $\mathbb{R}^n$. Fungsi tersebut merupakan bentuk kuadratik.

Berikut adalah beberapa definisi dan lemma dari bentuk kuadratik dan matriks definit positif:

Definisi 2.1

Misalkan $\mathbf{x}^T\mathbf{A}\mathbf{x}$ merupakan bentuk kuadratik dengan $\mathbf{A} = [a_{ij}]$ adalah matriks berukuran nxn dan vektor $\mathbf{x}^T = (\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_n)$ di $\mathbb{R}^n$. Bentuk kuadratik dikatakan definit positif jika:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0, \text{untuk setiap } \mathbf{x} \neq \mathbf{0}$$

(Howard Anton, 2000).

Definisi 2.2

Matriks simetris $\mathbf{A}$ disebut matriks definit positif jika $\mathbf{x}^T\mathbf{A}\mathbf{x}$ adalah bentuk kuadratik definit positif (Howard Anton, 2000).

Lemma 2.1

Sembarang matriks definit positif adalah matriks *nonsingular* (bukti diberikan pada Lampiran 1) (Howard Anton, 2000).

2.4 Matriks Partisi

Setiap matriks dapat dipartisi menjadi matriks-matriks yang lebih kecil. Matriks yang lebih kecil itu disebut submatriks. Berikut contoh matriks partisi yang mungkin dibuat dari matriks **A** berukuran $4x3$.

$$\mathbf{A} = \left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\mathbf{A} = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ \hline a_{41} & a_{42} & a_{43} \end{array}\right] = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ A_{41} \end{bmatrix}$$

Selanjutnya, untuk mencari invers dari suatu matriks partisi apabila submatriks dari matriks partisi merupakan submatriks yang *nonsingular*, maka dapat digunakan teorema berikut (bukti diberikan pada Lampiran 2):

Teorema 2.1

Misalkan matriks **M** berukuran $(p+q)x(p+q)$ yang dipartisi menjadi $\mathbf{M} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ dengan ukuran submatriks $\mathbf{A}_{\text{pxp}}$, $\mathbf{B}_{\text{pxq}}$, $\mathbf{C}_{\text{qxp}}$, $\mathbf{D}_{\text{qxq}}$ dan **A** *nonsingular* ($\mathbf{A}^{-1}$ada). **M** *nonsingular* jika dan hanya jika $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ *nonsingular*. Bentuk $\mathbf{M}^{-1}$ sebagai berikut:

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}' \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{E}^{-1} \end{bmatrix}$$

dengan $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ (Magnus dan Neudecker, 1999).

2.5 Differensiasi Matriks

Berikut adalah definisi, teorema dan lemma yang berkaitan dengan differensiasi pada matriks. Bukti-bukti dari teorema dan lemma tersebut, diberikan pada Lampiran 3.

Teorema 2.2

Jika $\mathbf{a}^{\mathrm{T}} = [\mathrm{a}_1 \quad \mathrm{a}_2 \quad \dots \quad \mathrm{a}_\mathrm{n}]$ adalah vektor baris dan

$$\mathbf{x} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \vdots \\ \mathrm{x}_\mathrm{n} \end{bmatrix}$$

maka

$$\frac{\partial(\mathbf{a}^{\mathrm{T}}\mathbf{x})}{\partial \mathbf{x}} = \mathbf{a} = \begin{bmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \\ \vdots \\ \mathrm{a}_\mathrm{n} \end{bmatrix}$$

(Magnus dan Neudecker, 1999).

Teorema 2.3

Misalkan $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ adalah

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \dots \quad \mathrm{x}_\mathrm{n}] \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \dots & \mathrm{a}_{1\mathrm{n}} \\ \vdots & \vdots & & \vdots \\ \mathrm{a}_{\mathrm{n}1} & \mathrm{a}_{\mathrm{n}2} & \dots & \mathrm{a}_{\mathrm{nn}} \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \vdots \\ \mathrm{x}_\mathrm{n} \end{bmatrix}$$

dimana **A** adalah matriks simetris. Maka

$$\frac{\partial(\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x})}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

(Magnus dan Neudecker, 1999).

Definisi 2.3

Misalkan **A** adalah matriks nxn, yang elemen pada baris ke$-i$ dan kolom ke$-k$ nya merupakan fungsi dari vektor parameter $\mathbf{x} = (x_1, \dots, x_n)^T$, dimana $i = 1, \dots, n\,; k = 1, \dots, n$. Turunan parsial pertama dari matriks **A** terhadap suatu skalar parameter x_j, dimana $j = 1, \dots, n$ adalah sebagai berikut:

$$\frac{\partial \mathbf{A}}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{a}_{11}}{\partial x_j} & \frac{\partial \mathrm{a}_{12}}{\partial x_j} & \cdots & \frac{\partial \mathrm{a}_{1n}}{\partial x_j} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \mathrm{a}_{n1}}{\partial x_j} & \frac{\partial \mathrm{a}_{n2}}{\partial x_j} & \cdots & \frac{\partial \mathrm{a}_{nn}}{\partial x_j} \end{bmatrix}$$

(Barnes, 2006).

Lemma 2.2

Misalkan *a* dan *b* adalah suatu konstanta. Bentuk matriks $\frac{\partial \mathbf{A}}{\partial x_j}$ dan $\frac{\partial \mathbf{B}}{\partial x_j}$, sesuai dengan Definisi 2.3 adalah sebagai berikut:

$$\frac{\partial \mathbf{A}}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{a}_{11}}{\partial x_j} & \frac{\partial \mathrm{a}_{12}}{\partial x_j} & \cdots & \frac{\partial \mathrm{a}_{1n}}{\partial x_j} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \mathrm{a}_{n1}}{\partial x_j} & \frac{\partial \mathrm{a}_{n2}}{\partial x_j} & \cdots & \frac{\partial \mathrm{a}_{nn}}{\partial x_j} \end{bmatrix}$$

dan

$$\frac{\partial \mathbf{B}}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{b}_{11}}{\partial x_j} & \frac{\partial \mathrm{b}_{12}}{\partial x_j} & \cdots & \frac{\partial \mathrm{b}_{1n}}{\partial x_j} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \mathrm{b}_{n1}}{\partial x_j} & \frac{\partial \mathrm{b}_{n2}}{\partial x_j} & \cdots & \frac{\partial \mathrm{b}_{nn}}{\partial x_j} \end{bmatrix}$$

maka

$$\frac{\partial (a\mathbf{A} + b\mathbf{B})}{\partial x_j} = a\frac{\partial \mathbf{A}}{\partial x_j} + b\frac{\partial \mathbf{B}}{\partial x_j}$$

dan

$$\frac{\partial (\mathbf{AB})}{\partial x_j} = \mathbf{A}\frac{\partial \mathbf{B}}{\partial x_j} + \frac{\partial \mathbf{A}}{\partial x_j}\mathbf{B}$$

(Barnes, 2006).

Lemma 2.3

Misalkan $\mathbf{A}$ matriks *nonsingular* ($\mathbf{A}^{-1}$ ada), berukuran nxn, yang elemen-elemennya adalah fungsi dari vektor parameter $\mathbf{x} = (x_1, \dots, x_n)^T$, sesuai dengan Definisi 2.3. Maka turunan parsial pertama dan kedua dari $\mathbf{A}^{-1}$ terhadap x_j, dimana $j = 1, \dots, n$ adalah

$$\frac{\partial \mathbf{A}^{-1}}{\partial x_j} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1}$$

dan

$$\frac{\partial^2 \mathbf{A}^{-1}}{\partial x_i \partial x_j} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i \partial x_j}\mathbf{A}^{-1} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}.$$

(Barnes, 2006).

Lemma 2.4

Misalkan $\mathbf{A}$ adalah matriks nxn yang elemen-elemennya adalah fungsi dari $\mathbf{x} = (x_1, \dots, x_n)^T$, sesuai dengan Definisi 2.3. Maka turunan parsial pertama dari $\det(\mathbf{A})$ terhadap skalar parameter x_j, dimana $j = 1, \dots, n$ adalah

$$\frac{\partial \det(\mathbf{A})}{\partial x_j} = tr\left[adj(\mathbf{A})\frac{\partial \mathbf{A}}{\partial x_j}\right].$$

(Barnes, 2006).

Lemma 2.5

Misalkan $\mathbf{A}$ adalah matriks nxn yang elemen-elemennya merupakan fungsi dari $\mathbf{x} = (x_1, \dots, x_n)^T$, sesuai dengan Definisi 2.3 dan $\det(\mathbf{A})$ adalah determinan dari matriks $\mathbf{A}$. Maka turunan parsial pertama dan kedua dari log $\det(\mathbf{A})$ terhadap skalar parameter x_j, dimana $j = 1, \dots, n$ adalah

$$\frac{\partial \log \det(\mathbf{A})}{\partial x_j} = tr\left[\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\right]$$

dan

$$\frac{\partial^2 \log \det(\mathbf{A})}{\partial x_i \partial x_j} = tr\left(\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i \partial x_j}\right) - tr\left(\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\right).$$

(Barnes, 2006).

2.6 Definisi *Translation-Invariant* dan *Maximal-Invariant*

Sebuah fungsi $s(\mathbf{y})$ dikatakan *translation-invariant* jika $s\left(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) = s(\mathbf{y})$ untuk setiap $\mathbf{y}$ dan $\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})$.

Jika diberikan $\widehat{\boldsymbol{\delta}}$ *translation-invariant* dan distribusi bersama dari $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ dan $\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ adalah multivariat normal, maka $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ independen terhadap

$\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ dan juga independen terhadap setiap fungsi dari $\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ (Kackar and Harville, 1984).

Misalkan χ suatu ruang vektor dan $\mathcal{G}$ adalah himpunan dari transformasi di χ. Fungsi $T(\mathbf{x})$ pada χ dikatakan *maximal-invariant* pada $\mathcal{G}$ jika:

i. $T\big(g(\mathbf{x})\big) = T(\mathbf{x})$, untuk setiap $\mathbf{x} \in \chi$ dan $g \in \mathcal{G}$; (invariant)

ii. $T(\mathbf{x}_1) = T(\mathbf{x}_2) \rightarrow \mathbf{x}_1 = g(\mathbf{x}_2)$, untuk setiap $g \in \mathcal{G}$.

2.7 *General Linear Mixed Model* (GLMM)

Sebelum membahas GLMM, terlebih dahulu akan dijelaskan mengenai model regresi linear berganda. Pandang bentuk model regresi linear berganda (Montgomery, Peck, & Vining, 2001) berikut ini:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

dimana

$\mathbf{y}$: vektor dari variabel dependen, berukuran $nx1$

$\mathbf{X}$: matriks dari variabel prediktor, berukuran nxp, dimana $p = k + 1$

$\boldsymbol{\beta}$: vektor dari parameter, berukuran $px1$

$\boldsymbol{\varepsilon}$: vektor dari *error*, berukuran $nx1$, dengan $\varepsilon_i \sim$ NID $(0, \sigma^2)$.

Pada model regresi linear tersebut, pengaruh tetap yang terkandung di dalam model telah diperhatikan, yaitu **X**. Namun, pengaruh acak, yaitu **v**, tidak terkandung di dalam model. Sedangkan bentuk *General Linear Mixed Model* adalah sebagai berikut:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e} \qquad (2.2)$$

dimana

$\mathbf{y}$: vektor dari variabel respon, bersifat *random*, berukuran $nx1$

$\mathbf{X}$: matriks *full rank* dari variabel prediktor yang diketahui, berukuran nxp

$\boldsymbol{\beta}$: vektor dari parameter, bersifat *fixed*, berukuran $px1$

$\mathbf{Z}$: matriks *full rank* dari variabel prediktor yang diketahui, berukuran nxh

$\mathbf{v}$: vektor dari parameter, bersifat *random*, berukuran $hx1$

$\mathbf{e}$: vektor dari *error*, bersifat *random*, berukuran $nx1$

dengan asumsi:

$$E(\mathbf{v}) = \mathbf{0} \text{ , } E(\mathbf{e}) = \mathbf{0},$$

$$\mathbf{v} \text{ dan } \mathbf{e} \text{ saling independen,}$$

$$E(\mathbf{v}\mathbf{v}^{\boldsymbol{T}}) = \mathbf{G},\ E(\mathbf{e}\mathbf{e}^{\boldsymbol{T}}) = \mathbf{R}. \qquad (2.3)$$

Karena **v** dan **e** saling independen, maka

$$\begin{aligned} cov(\mathbf{v},\mathbf{e}) &= cov(\mathbf{e},\mathbf{v}) \\ &= E(\mathbf{v}-\mathbf{0})(\mathbf{e}-\mathbf{0})^T \\ &= E(\mathbf{e}-\mathbf{0})(\mathbf{v}-\mathbf{0})^T = \mathbf{0} \end{aligned}$$

sehingga $corr(\mathbf{v},\mathbf{e}) = \mathbf{0}$. **G** dan **R** masing-masing adalah matriks varians-kovarian dari **v** dan **e** yang elemen-elemennya merupakan fungsi dalam $\boldsymbol{\delta} = \left(\delta_1, \delta_2, \dots, \delta_q\right)^T$. $\boldsymbol{\delta}$ adalah vektor parameter dari variansi **v** dan **e**.

Bentuk dari matriks **G** dan **R** yaitu:

$$\mathbf{G}(\boldsymbol{\delta}) = \begin{pmatrix} g_{11}(\boldsymbol{\delta}) & g_{12}(\boldsymbol{\delta}) & \cdots & g_{1h}(\boldsymbol{\delta}) \\ g_{21}(\boldsymbol{\delta}) & g_{22}(\boldsymbol{\delta}) & \dots & g_{2h}(\boldsymbol{\delta}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{h1}(\boldsymbol{\delta}) & g_{h2}(\boldsymbol{\delta}) & \cdots & g_{hh}(\boldsymbol{\delta}) \end{pmatrix}$$

dan (2.4)

$$\mathbf{R}(\boldsymbol{\delta}) = \begin{pmatrix} r_{11}(\boldsymbol{\delta}) & r_{12}(\boldsymbol{\delta}) & \cdots & r_{1n}(\boldsymbol{\delta}) \\ r_{21}(\boldsymbol{\delta}) & r_{22}(\boldsymbol{\delta}) & \dots & r_{2n}(\boldsymbol{\delta}) \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1}(\boldsymbol{\delta}) & r_{n2}(\boldsymbol{\delta}) & \cdots & r_{nn}(\boldsymbol{\delta}) \end{pmatrix}$$

Sedangkan $E(\mathbf{y})$ dan $var(\mathbf{y})$ didapat dari:

$$\begin{aligned} E(\mathbf{y}) &= E(\mathbf{X\beta} + \mathbf{Zv} + \mathbf{e}) \\ &= E(\mathbf{X\beta}) + E(\mathbf{Zv}) + E(\mathbf{e}) \\ &= \mathbf{X\beta} + \mathbf{Z}E(\mathbf{v}) + E(\mathbf{e}) \\ &= \mathbf{X\beta}\ ; \text{karena } E(\mathbf{v}) = \mathbf{0} \text{ dan } E(\mathbf{e}) = \mathbf{0} \qquad (2.5) \end{aligned}$$

$$\begin{aligned} var(\mathbf{y}) &= E((\mathbf{y}-\mathbf{X\beta})(\mathbf{y}-\mathbf{X\beta})^T) \\ &= E\big((\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\big) \\ &= E\Big((\mathbf{Zv}+\mathbf{e})(\mathbf{v}^{\boldsymbol{T}}\mathbf{Z}^{\boldsymbol{T}}+\mathbf{e}^{\boldsymbol{T}})\Big) \\ &= E(\mathbf{Zvv}^{\boldsymbol{T}}\mathbf{Z}^{\boldsymbol{T}} + \mathbf{Zve}^{\boldsymbol{T}} + \mathbf{ev}^{\boldsymbol{T}}\mathbf{Z}^{\boldsymbol{T}} + \mathbf{ee}^{\boldsymbol{T}}) \\ &= E(\mathbf{Zvv}^{\boldsymbol{T}}\mathbf{Z}^{\boldsymbol{T}}) + E(\mathbf{Zve}^{\boldsymbol{T}}) + E(\mathbf{ev}^{\boldsymbol{T}}\mathbf{Z}^{\boldsymbol{T}}) + E(\mathbf{ee}^{\boldsymbol{T}}) \\ &= \mathbf{Z}E(\mathbf{vv}^{\boldsymbol{T}})\mathbf{Z}^{\boldsymbol{T}} + \mathbf{Z}E(\mathbf{ve}^{\boldsymbol{T}}) + E(\mathbf{ev}^{\boldsymbol{T}})\mathbf{Z}^{\boldsymbol{T}} + \mathbf{R} \qquad ; \text{karena } E(\mathbf{ee}^{\boldsymbol{T}}) = \mathbf{R} \\ &= \mathbf{ZGZ}^{\boldsymbol{T}} + \mathbf{0} + \mathbf{0} + \mathbf{R} \qquad ; \text{karena } \mathbf{v} \text{ dan } \mathbf{e} \text{ saling independen} \\ &= \mathbf{ZGZ}^{\boldsymbol{T}} + \mathbf{R} = \mathbf{V}(\boldsymbol{\delta}) \qquad (2.6) \end{aligned}$$

dengan $\mathbf{V}(\boldsymbol{\delta})$ menyatakan matriks varians-kovarian dari **y** (Rao, 2003).

2.8 Metode BLUP dan MSE BLUP pada GLMM

Pada metode *Best Linear Unbiased Prediction* (BLUP) diasumsikan bahwa nilai $\boldsymbol{\delta}$ diketahui. Penaksiran parameter dengan metode BLUP pada GLMM dapat dilakukan dengan beberapa tahapan. Pertama, dimisalkan nilai-nilai dari $\mathbf{v} = \begin{pmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \vdots \\ \mathrm{v}_\mathrm{h} \end{pmatrix}$ adalah $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_h \end{pmatrix}$. Dalam hal ini, yang akan ditaksir adalah kombinasi linear, $\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\boldsymbol{v}$, dengan $\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_p \end{pmatrix}$ dan $\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix}$.

Misalkan $T(\mathbf{y}) = \boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b$ adalah sembarang penaksir dari kombinasi linear $\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\boldsymbol{v}$, dimana $\boldsymbol{l} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix}$ dan b adalah suatu konstanta. $T(\mathbf{y})$ disebut penaksir BLUP yang dapat mengestimasi kombinasi linear tersebut jika $T(\mathbf{y})$ merupakan penaksir yang bersifat:

1. Linear,

$T(\mathbf{y})$ dikatakan linear jika $T(p\mathbf{y_1} + q\mathbf{y_2}) = pT(\mathbf{y_1}) + qT(\mathbf{y_2})$ (Howard Anton, 2000) dengan p dan q suatu skalar. Dengan $T(\mathbf{y}) = \boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b$, akan ditunjukkan bahwa $T(\mathbf{y})$ bersifat linear:

$$\begin{aligned} T(p\mathbf{y_1} + q\mathbf{y_2}) &= \boldsymbol{l}^{\boldsymbol{T}}(p\mathbf{y_1} + q\mathbf{y_2}) + b \\ &= \boldsymbol{l}^{\boldsymbol{T}}p\mathbf{y_1} + \boldsymbol{l}^{\boldsymbol{T}}q\mathbf{y_2} + pb_1 + qb_2; \text{ dengan } b_1 = \tfrac{b}{2p} \text{ dan } b_2 = \tfrac{b}{2q} \\ &= p(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{y_1}) + pb_1 + q(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{y_2}) + qb_2 \\ &= p(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b_1) + q(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b_2) \\ &= pT(\mathbf{y_1}) + qT(\mathbf{y_2}) \end{aligned} \tag{2.7}$$

2. *Unbiased*,

Definisi *unbiased* yang biasa digunakan adalah $E(T(\mathbf{y})) = \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}$. Namun, pada ruas kanan persamaan tersebut, masih terkandung vektor random $\mathbf{v}$, sehingga perlu dicari nilai ekspektasi dari $\mathbf{v}$. Oleh karena itu, definisi *unbiased* yang digunakan dalam hal ini adalah $E(T(\mathbf{y})) = E(\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v})$ (Rao, 2003). Berikut akan ditunjukkan bahwa $T(\mathbf{y})$ bersifat *unbiased*.

Diketahui: $T(\mathbf{y}) = \boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b$

Akan ditunjukkan: $E(T(\mathbf{y})) = E(\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v})$

$$\begin{aligned}
E(T(\mathbf{y})) &= E(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b) \\
&= E(\boldsymbol{l}^T\mathbf{X\beta} + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{Zv} + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{e}) + E(b) \\
&= E(\boldsymbol{l}^T\mathbf{X\beta}) + E(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{Zv}) + E(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{e}) + b \\
&= \boldsymbol{l}^T E(\mathbf{X\beta}) + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{Z}E(\mathbf{v}) + \boldsymbol{l}^{\boldsymbol{T}}E(\mathbf{e}) + b \\
&= \boldsymbol{l}^T\mathbf{X\beta} + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{Z}(\mathbf{0}) + \boldsymbol{l}^{\boldsymbol{T}}(\mathbf{0}) + b \text{ ; karena } E(\mathbf{v}) = \mathbf{0}\text{ , } E(\mathbf{e}) = \mathbf{0} \\
&= \boldsymbol{l}^T\mathbf{X\beta} + b \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
E(\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}) &= E(\boldsymbol{\lambda}^T\boldsymbol{\beta}) + E(\boldsymbol{\omega}^T\mathbf{v}) \\
&= \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T E(\mathbf{v}) \\
&= \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T(\mathbf{0}) \text{ ; karena } \boldsymbol{E}(\mathbf{v}) = \mathbf{0} \\
&= \boldsymbol{\lambda}^T\boldsymbol{\beta} \qquad (2.9)
\end{aligned}$$

Dari persamaan (2.8) dan (2.9) diperoleh $E(T(\mathbf{y})) \neq E(\boldsymbol{\lambda}^{\boldsymbol{T}}\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v})$, sehingga $T(\mathbf{y})$ bersifat *biased*. Namun, jika $b = 0$ dan $\boldsymbol{\lambda}^T$ kombinasi linear dari baris-baris **X**, yaitu $\boldsymbol{\lambda}^T = \boldsymbol{l}^T\mathbf{X}$, maka $T(\mathbf{y})$ bersifat *unbiased* terhadap $\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}$.

Berikut akan ditunjukkan bahwa jika $b = 0$ dan $\boldsymbol{\lambda}^T = \boldsymbol{l}^T\mathbf{X}$, maka $T(\mathbf{y}) = \boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b$ merupakan penaksir *unbiased* untuk $\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}$.

Diketahui: $b = 0$ dan $\boldsymbol{\lambda}^T = \boldsymbol{l}^T\mathbf{X}$,

Akan ditunjukkan: $E(T(\mathbf{y})) = E(\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v})$

$$\begin{aligned}
E\big(T(\mathbf{y})\big) &= E(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{y} + b) \\
&= E(\boldsymbol{l}^T(\mathbf{X\beta} + \mathbf{Zv} + \mathbf{e}) + 0) \text{ ; karena } b = 0. \\
&= E(\boldsymbol{l}^T\mathbf{X\beta} + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{Zv} + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{e}) \\
&= E(\boldsymbol{l}^T\mathbf{X\beta}) + E(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{Zv}) + E(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{e}) \\
&= \boldsymbol{l}^T\mathbf{X\beta} + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{Z}E(\mathbf{v}) + \boldsymbol{l}^{\boldsymbol{T}}E(\mathbf{e}) \\
&= \boldsymbol{l}^T\mathbf{X\beta} + \boldsymbol{l}^{\boldsymbol{T}}\mathbf{Z}(\mathbf{0}) + \boldsymbol{l}^{\boldsymbol{T}}(\mathbf{0}) \text{ ; karena } E(\mathbf{v}) = \mathbf{0} \text{ dan } E(\mathbf{e}) = \mathbf{0} \\
&= \boldsymbol{l}^T\mathbf{X\beta} \\
&= \boldsymbol{\lambda}^T\boldsymbol{\beta} \qquad \text{; karena } \boldsymbol{\lambda}^T = \boldsymbol{l}^T\mathbf{X}. \qquad (2.10)
\end{aligned}$$

Untuk selanjutnya, akan digunakan $T(\mathbf{y}) = \boldsymbol{l}^{\boldsymbol{T}}\mathbf{y}$ sebagai penaksir untuk $\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\boldsymbol{v}$.

3. Terbaik (*best*)

Penaksir $T(\mathbf{y})$ dikatakan bersifat *best* jika $T(\mathbf{y})$ memiliki MSE terkecil diantara penaksir *unbiased* dan linear lainnya (Rao, 2003). Ambil sembarang kombinasi linear $\boldsymbol{\tau} = \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\boldsymbol{\nu}$ yang dapat diestimasi oleh $T(\mathbf{y}) = \boldsymbol{l}^T\mathbf{y}$. Selain itu, didefinisikan juga $E[(T(\mathbf{y}) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}))^2]$ sebagai MSE dari $T(\mathbf{y})$.

Selanjutnya, akan ditunjukkan bahwa $E[(T(\mathbf{y}) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}))^2]$ minimum terhadap kendala $\boldsymbol{l}^T\mathbf{X} - \boldsymbol{\lambda}^T = \mathbf{0}$. Kemudian akan digunakan metode *Lagrange*, untuk meminimumkan MSE dari $T(\mathbf{y})$ terhadap satu kendala (Robinson, 1991). Namun, sebelumnya akan didefinisikan fungsi-fungsi berikut:

$$\begin{aligned} T(\mathbf{y}) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}) &= \boldsymbol{l}^T(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e}) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}) \\ &= \boldsymbol{l}^T\mathbf{X}\boldsymbol{\beta} + \boldsymbol{l}^T\mathbf{Z}\mathbf{v} + \boldsymbol{l}^T\mathbf{e} - \boldsymbol{\lambda}^T\boldsymbol{\beta} - \boldsymbol{\omega}^T\mathbf{v} \\ &= (\boldsymbol{l}^T\mathbf{X} - \boldsymbol{\lambda}^T)\boldsymbol{\beta} + \boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^T\mathbf{v} \\ &= (\mathbf{0})\boldsymbol{\beta} + \boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^T\mathbf{v} \text{ ; karena } \boldsymbol{l}^T\mathbf{X} - \boldsymbol{\lambda}^T = \mathbf{0} \\ &= \boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^T\mathbf{v} \qquad (2.11) \end{aligned}$$

$$\begin{aligned} (\boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^T\mathbf{v})^T &= (\boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}))^T - (\boldsymbol{\omega}^T\mathbf{v})^T \\ &= (\mathbf{Z}\mathbf{v} + \mathbf{e})^T\boldsymbol{l} - \mathbf{v}^T\boldsymbol{\omega} \\ &= (\mathbf{v}^T\mathbf{Z}^T + \mathbf{e}^T)\boldsymbol{l} - \mathbf{v}^T\boldsymbol{\omega} \qquad (2.12) \end{aligned}$$

Dari persamaan (2.11) dan (2.12), kemudian akan diturunkan bentuk MSE dari $T(\mathbf{y})$:

$$\begin{aligned} \text{MSE } [T(\mathbf{y})] &= E[(T(\mathbf{y}) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}))^2] \\ &= E[(T(\mathbf{y}) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}))(T(\mathbf{y}) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}))^T] \\ &= E[(\boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^T\mathbf{v})(\boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^T\mathbf{v})^T] \text{ ; dari (2.11)} \\ &= E[(\boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^T\mathbf{v})((\mathbf{v}^T\mathbf{Z}^T + \mathbf{e}^T)\boldsymbol{l} - \mathbf{v}^T\boldsymbol{\omega})] \text{ ; dari (2.12)} \\ &= E[\boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e})(\mathbf{v}^T\mathbf{Z}^T + \mathbf{e}^T)\boldsymbol{l} - \boldsymbol{l}^T(\mathbf{Z}\mathbf{v} + \mathbf{e})\mathbf{v}^T\boldsymbol{\omega} \\ &\quad -\boldsymbol{\omega}^T\mathbf{v}(\mathbf{v}^T\mathbf{Z}^T + \mathbf{e}^T)\boldsymbol{l} + \boldsymbol{\omega}^T\mathbf{v}\mathbf{v}^T\boldsymbol{\omega}] \\ &= \underbrace{E[(\boldsymbol{l}^T\mathbf{Z}\mathbf{v} + \boldsymbol{l}^T\mathbf{e})(\mathbf{v}^T\mathbf{Z}^T\boldsymbol{l} + \mathbf{e}^T\boldsymbol{l})]}_{(1)} - \underbrace{E[(\boldsymbol{l}^T\mathbf{Z}\mathbf{v} + \boldsymbol{l}^T\mathbf{e})\mathbf{v}^T\boldsymbol{\omega}]}_{(2)} \\ &\quad - \underbrace{E[\boldsymbol{\omega}^T\mathbf{v}(\mathbf{v}^T\mathbf{Z}^T\boldsymbol{l} + \mathbf{e}^T\boldsymbol{l})]}_{(3)} - \underbrace{E[\boldsymbol{\omega}^T\mathbf{v}\mathbf{v}^T\boldsymbol{\omega}]}_{(4)} \end{aligned}$$

Berikut akan dijabarkan masing-masing nilai dari (1), (2), (3), dan (4).

(1) $E[(\boldsymbol{l}^T\mathbf{Z}\mathbf{v} + \boldsymbol{l}^T\mathbf{e})(\mathbf{v}^T\mathbf{Z}^T\boldsymbol{l} + \mathbf{e}^T\boldsymbol{l})]$

$$= E[\boldsymbol{l}^T\mathbf{Z}\mathbf{v}\mathbf{v}^T\mathbf{Z}^T\boldsymbol{l} + \boldsymbol{l}^T\mathbf{Z}\mathbf{v}\mathbf{e}^T\boldsymbol{l} + \boldsymbol{l}^T\mathbf{e}\mathbf{v}^T\mathbf{Z}^T\boldsymbol{l} + \boldsymbol{l}^T\mathbf{e}\mathbf{e}^T\boldsymbol{l}]$$

$= E[\boldsymbol{l}^T \mathbf{Z}\mathbf{v}\mathbf{v}^T \mathbf{Z}^T \boldsymbol{l}] + E[\boldsymbol{l}^T \mathbf{Z}\mathbf{v}\mathbf{e}^T \boldsymbol{l}] + E[\boldsymbol{l}^T \mathbf{e}\mathbf{v}^T \mathbf{Z}^T \boldsymbol{l}] + E[\boldsymbol{l}^T \mathbf{e}\mathbf{e}^T \boldsymbol{l}]$

$= \boldsymbol{l}^T \mathbf{Z} E(\boldsymbol{v}\mathbf{v}^T)\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{Z} E(\boldsymbol{v}\mathbf{e}^T)\boldsymbol{l} + \boldsymbol{l}^T E(\mathbf{e}\mathbf{v}^T)\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T E(\mathbf{e}\mathbf{e}^T)\boldsymbol{l}$

$= \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{Z}(\mathbf{0})\boldsymbol{l} + \boldsymbol{l}^T(\mathbf{0})\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R}\boldsymbol{l}$

$= \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R}\boldsymbol{l}$

(2) $E[(\boldsymbol{l}^T \mathbf{Z}\mathbf{v} + \boldsymbol{l}^T \mathbf{e})\mathbf{v}^T \boldsymbol{\omega}]$

$= E[\boldsymbol{l}^T \mathbf{Z}\mathbf{v}\mathbf{v}^T \boldsymbol{\omega} + \boldsymbol{l}^T \mathbf{e}\mathbf{v}^T \boldsymbol{\omega}]$

$= \boldsymbol{l}^T \mathbf{Z} E(\boldsymbol{v}\mathbf{v}^T)\boldsymbol{\omega} + \boldsymbol{l}^T E(\mathbf{e}\mathbf{v}^T)\boldsymbol{\omega}$

$= \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\boldsymbol{\omega} + \boldsymbol{l}^T(\mathbf{0})\boldsymbol{\omega}$

$= \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\boldsymbol{\omega}$

(3) $E[\boldsymbol{\omega}^T \mathbf{v}(\mathbf{v}^T \mathbf{Z}^T \boldsymbol{l} + \mathbf{e}^T \boldsymbol{l})]$

$= E[\boldsymbol{\omega}^T \mathbf{v}\mathbf{v}^T \mathbf{Z}^T \boldsymbol{l} + \boldsymbol{\omega}^T \mathbf{v}\mathbf{e}^T \boldsymbol{l}]$

$= \boldsymbol{\omega}^T E(\mathbf{v}\mathbf{v}^T)\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{\omega}^T E(\mathbf{v}\mathbf{e}^T)\boldsymbol{l}$

$= \boldsymbol{\omega}^T \mathbf{G}\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{\omega}^T(\mathbf{0})\boldsymbol{l}$

$= \boldsymbol{\omega}^T \mathbf{G}\mathbf{Z}^T \boldsymbol{l}$

(4) $E[\boldsymbol{\omega}^T \mathbf{v}\mathbf{v}^T \boldsymbol{\omega}]$

$= \boldsymbol{\omega}^T E(\mathbf{v}\mathbf{v}^T)\boldsymbol{\omega}$

$= \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega}$

Dari (1), (2), (3), dan (4) didapat:

$$MSE\ [T(\mathbf{y})] = \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R}\boldsymbol{l} - \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\boldsymbol{\omega} - \boldsymbol{\omega}^T \mathbf{G}\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega}$$

$$= \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R}\boldsymbol{l} - 2\boldsymbol{l}^T \mathbf{Z}\mathbf{G}\boldsymbol{\omega} + \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega}\ ; \quad (2.13)$$

karena $(\boldsymbol{\omega}^T \mathbf{G}\mathbf{Z}^T \boldsymbol{l})^T = \boldsymbol{l}^T \boldsymbol{Z}\mathbf{G}\boldsymbol{\omega}$ dan $\mathbf{G}$ simetris.

Kemudian dengan metode *Lagrange* yang telah dibahas pada subbab sebelumnya, dibentuk fungsi *Lagrange* berikut:

$$f = \boldsymbol{l}^T \mathbf{Z}\mathbf{G}\mathbf{Z}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R}\boldsymbol{l} - 2\boldsymbol{l}^T \mathbf{Z}\mathbf{G}\boldsymbol{\omega} + \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega} - 2\boldsymbol{\eta}^T(\mathbf{X}^T \boldsymbol{l} - \boldsymbol{\lambda}) \quad (2.14)$$

Fungsi *Lagrange* tersebut akan diturunkan terhadap $\boldsymbol{l}$ dan $\boldsymbol{\eta}$, dengan

$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_p \end{pmatrix}$ adalah pengali *Lagrange*, sehingga akan didapat nilai $\boldsymbol{l}$ dan $\boldsymbol{\eta}$ yang meminimumkan fungsi *f*.

Berikut ini turunan parsial dari f terhadap $\boldsymbol{l}$ dan $\boldsymbol{\eta}$.

$$f = \boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R} \boldsymbol{l} - 2\boldsymbol{l}^T \mathbf{ZG}\boldsymbol{\omega} + \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega} - 2\boldsymbol{\eta}^T(\mathbf{X}^T \boldsymbol{l} - \boldsymbol{\lambda})$$

$$= \boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R} \boldsymbol{l} - 2\boldsymbol{l}^T \mathbf{ZG}\boldsymbol{\omega} + \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega} - 2\boldsymbol{\eta}^T \mathbf{X}^T \boldsymbol{l} + 2\boldsymbol{\eta}^T \boldsymbol{\lambda}$$

dengan

$$\frac{\partial f}{\partial \boldsymbol{l}} = \frac{\partial(\boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R} \boldsymbol{l} - 2\boldsymbol{l}^T \mathbf{ZG}\boldsymbol{\omega} + \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega} - 2\boldsymbol{\eta}^T \mathbf{X}^T \boldsymbol{l} + 2\boldsymbol{\eta}^T \boldsymbol{\lambda})}{\partial \boldsymbol{l}}$$

$$= \frac{\partial(\boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l}) + \partial(\boldsymbol{l}^T \mathbf{R} \boldsymbol{l}) - \partial(2\boldsymbol{l}^T \mathbf{ZG}\boldsymbol{\omega}) + \partial(\boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega}) - \partial(2\boldsymbol{\eta}^T \mathbf{X}^T \boldsymbol{l}) + \partial(2\boldsymbol{\eta}^T \boldsymbol{\lambda})}{\partial \boldsymbol{l}}$$

$$= \frac{\partial(\boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l})}{\partial \boldsymbol{l}} + \frac{\partial(\boldsymbol{l}^T \mathbf{R} \boldsymbol{l})}{\partial \boldsymbol{l}} - \frac{\partial(2\boldsymbol{l}^T \mathbf{ZG}\boldsymbol{\omega})}{\partial \boldsymbol{l}} + \frac{\partial(\boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega})}{\partial \boldsymbol{l}} - \frac{\partial(2\boldsymbol{\eta}^T \mathbf{X}^T \boldsymbol{l})}{\partial \boldsymbol{l}} + \frac{\partial(2\boldsymbol{\eta}^T \boldsymbol{\lambda})}{\partial \boldsymbol{l}}$$

$$= 2\mathbf{ZGZ}^T \boldsymbol{l} + 2\mathbf{R}\boldsymbol{l} - 2\mathbf{ZG}\boldsymbol{\omega} + \mathbf{0} - 2\mathbf{X}\boldsymbol{\eta} + \mathbf{0}$$

$$= 2\mathbf{ZGZ}^T \boldsymbol{l} + 2\mathbf{R}\boldsymbol{l} - 2\mathbf{ZG}\boldsymbol{\omega} - 2\mathbf{X}\eta$$

dan

$$\frac{\partial f}{\partial \boldsymbol{\eta}} = \frac{\partial(\boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l} + \boldsymbol{l}^T \mathbf{R} \boldsymbol{l} - 2l^T \mathbf{ZG}\omega + \boldsymbol{\omega}^T \mathbf{G}\boldsymbol{\omega} - 2\boldsymbol{\eta}^T \mathbf{X}^T \boldsymbol{l} + 2\boldsymbol{\eta}^T \lambda)}{\partial \boldsymbol{\eta}}$$

$$= \frac{\partial(\boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l}) + \partial(l^T \mathbf{R} l) - \partial(2l^T \mathbf{ZG}\omega) + \partial(\boldsymbol{\omega}^T \mathbf{G}\omega) - \partial(2\boldsymbol{\eta}^T \mathbf{X}^T \boldsymbol{l}) + \partial(2\boldsymbol{\eta}^T \boldsymbol{\lambda})}{\partial \boldsymbol{\eta}}$$

$$= \frac{\partial(\boldsymbol{l}^T \mathbf{ZGZ}^T \boldsymbol{l})}{\partial \boldsymbol{\eta}} + \frac{\partial(\boldsymbol{l}^T \mathbf{R} l)}{\partial \boldsymbol{\eta}} - \frac{\partial(2\boldsymbol{l}^T \mathbf{ZG}\boldsymbol{\omega})}{\partial \boldsymbol{\eta}} + \frac{\partial(\boldsymbol{\omega}^T \mathbf{G}\omega)}{\partial \boldsymbol{\eta}} - \frac{\partial(2\boldsymbol{\eta}^T \mathbf{X}^T \boldsymbol{l})}{\partial \boldsymbol{\eta}} + \frac{\partial(2\boldsymbol{\eta}^T \boldsymbol{\lambda})}{\partial \boldsymbol{\eta}}$$

$$= \mathbf{0} + \mathbf{0} - \mathbf{0} + \mathbf{0} - 2\mathbf{X}^T \boldsymbol{l} + 2\lambda$$

$$= -2\mathbf{X}^T \boldsymbol{l} + 2\lambda$$

sehingga diperoleh:

$$\frac{\partial f}{\partial \boldsymbol{l}} = 2\mathbf{ZGZ}^T \boldsymbol{l} + 2\mathbf{R}\boldsymbol{l} - 2\mathbf{ZG}\omega - 2\mathbf{X}\boldsymbol{\eta}$$

$$\frac{\partial f}{\partial \boldsymbol{\eta}} = -2\mathbf{X}^T \boldsymbol{l} + 2\lambda$$

Berdasarkan metode *Lagrange*, nilai $\boldsymbol{l}$ dan $\boldsymbol{\eta}$ dapat diperoleh dari:

$$\frac{\partial f}{\partial \boldsymbol{l}} = 2(\mathbf{ZGZ}^T + \mathbf{R})\boldsymbol{l} - 2\mathbf{X}\boldsymbol{\eta} = 2\mathbf{ZG}\boldsymbol{\omega}$$

$$\frac{\partial f}{\partial \boldsymbol{\eta}} = -2\mathbf{X}^T \boldsymbol{l} + 2\boldsymbol{\lambda} = \mathbf{0}$$

dengan $\mathbf{ZGZ}^T + \mathbf{R} = \mathbf{V} = \mathbf{V}(\boldsymbol{\delta})$ didapat persamaan:

$$\mathbf{V}\boldsymbol{l} - \mathbf{X}\boldsymbol{\eta} = \mathbf{ZG}\boldsymbol{\omega}$$

$$\mathbf{X}^T \boldsymbol{l} - \boldsymbol{\lambda} = \mathbf{0}$$

kemudian dibentuk menjadi persamaan:

$$\begin{bmatrix} \mathbf{V}(\boldsymbol{\delta}) & \mathbf{X} \\ \mathbf{X}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{l}} \\ -\hat{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathbf{ZG}\boldsymbol{\omega} \\ \boldsymbol{\lambda} \end{bmatrix}$$

Matriks $\mathbf{G}$, $\mathbf{R}$, dan $\mathbf{V}$ adalah matriks definit positif. Berdasarkan Lemma 2.1 pada subbab sebelumnya, sembarang matriks definit positif merupakan matriks *nonsingular*. Untuk mencari nilai $\hat{\boldsymbol{l}}$ dan $\hat{\boldsymbol{\eta}}$, dapat digunakan teorema invers untuk matriks partisi yang telah dijelaskan pada subbab 2.4, sehingga didapat:

$$\begin{bmatrix} \hat{\boldsymbol{l}} \\ -\hat{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{-1}(\delta)[\mathbf{I} - \mathbf{X}(\mathbf{X}^{-1}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}] & \mathbf{V}^{-1}(\delta)\mathbf{X}(\mathbf{X}^{-1}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1} \\ (\mathbf{X}^{T}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\delta) & -(\mathbf{X}^{T}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{ZG}\boldsymbol{\omega} \\ \boldsymbol{\lambda} \end{bmatrix}$$

dengan

$$\hat{\boldsymbol{l}} = \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}^{-1}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\boldsymbol{\lambda} + \mathbf{V}^{-1}(\delta)[\mathbf{I} - \mathbf{X}(\mathbf{X}^{-1}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\delta)]\mathbf{ZG}\boldsymbol{\omega}$$

$$-\hat{\boldsymbol{\eta}} = (\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{ZG}\boldsymbol{\omega} - (\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\boldsymbol{\lambda} \qquad (2.15)$$

Karena sembarang penaksir $T(\mathbf{y}) = \boldsymbol{l}^{T}\mathbf{y}$ memenuhi sifat *best*, linear, dan *unbiased*, maka:

$$T(\mathbf{y}) = \boldsymbol{l}^{T}\mathbf{y}$$

$$= \{\mathbf{V}^{-1}(\delta)\mathbf{X}(\mathbf{X}^{-1}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\lambda + \mathbf{V}^{-1}(\delta)[\mathbf{I} - \mathbf{X}(\mathbf{X}^{-1}\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\delta)]\mathbf{ZG}\omega\}^{T}\mathbf{y}$$

karena $\mathbf{G}$ dan $\mathbf{V}^{-1}(\boldsymbol{\delta})$ simetris, maka:

$$= \boldsymbol{\lambda}^{T}\underbrace{(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y}}_{\hat{\boldsymbol{\beta}}} + \boldsymbol{\omega}^{T}\mathbf{GZ}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y}$$

$$-\boldsymbol{\omega}^{T}\mathbf{GZ}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\underbrace{(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y}}_{\hat{\boldsymbol{\beta}}}$$

dengan $(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y} = \hat{\boldsymbol{\beta}}$, persamaan sebelumnya dapat ditulis:

$$= \boldsymbol{\lambda}^{T}\hat{\boldsymbol{\beta}} + \boldsymbol{\omega}^{T}\mathbf{GZ}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y} - \boldsymbol{\omega}^{T}\mathbf{GZ}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\hat{\boldsymbol{\beta}}$$

$$= \boldsymbol{\lambda}^{T}\hat{\boldsymbol{\beta}} + \boldsymbol{\omega}^{T}\underbrace{\mathbf{GZ}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})}_{\hat{\boldsymbol{v}}}$$

sehingga didapat taksiran BLUP untuk $\boldsymbol{\beta}$ dan $\boldsymbol{v}$ sebagai berikut:

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}, \mathbf{y}) = (\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y}$$

dan (2.16)

$$\hat{\boldsymbol{v}}(\boldsymbol{\delta}, \mathbf{y}) = \mathbf{GZ}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}, \mathbf{y}))$$

dimana

$$\mathbf{V}(\boldsymbol{\delta}) = \mathbf{R} + \mathbf{ZGZ}^{T}.$$

Dengan demikian, taksiran BLUP dari $\boldsymbol{\tau} = \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\boldsymbol{v}$ adalah:

$$\hat{\boldsymbol{\tau}}(\boldsymbol{\delta},\mathbf{y}) = \boldsymbol{\lambda}^T\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) + \boldsymbol{\omega}^T\hat{\boldsymbol{v}}(\boldsymbol{\delta},\mathbf{y}) \tag{2.17}$$

yang memiliki MSE minimum dari semua taksiran yang *unbiased* dan linear.

Selanjutnya akan dicari nilai MSE BLUP untuk mengukur seberapa baik taksiran BLUP yang telah diperoleh. Namun, untuk mempermudah perhitungan, dengan melakukan sedikit modifikasi aljabar, taksiran BLUP yang diperoleh pada persamaan (2.16) dinyatakan dalam bentuk lain, yaitu:

$$\begin{aligned}
\hat{\boldsymbol{\tau}}(\boldsymbol{\delta},\mathbf{y}) &= \boldsymbol{\lambda}^T\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) + \boldsymbol{\omega}^T\hat{\boldsymbol{v}}(\boldsymbol{\delta},\mathbf{y})\\
&= \boldsymbol{\lambda}^T\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) + \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}))\\
&= \boldsymbol{\lambda}^T\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) + \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y} - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y})\\
&= \boldsymbol{\lambda}^T\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) + \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y} - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) + \boldsymbol{\lambda}^T\boldsymbol{\beta} - \boldsymbol{\lambda}^T\boldsymbol{\beta}\\
&\qquad + \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\boldsymbol{\beta} - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\boldsymbol{\beta}\\
&= \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y} - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\lambda}^T\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) - \boldsymbol{\lambda}^T\boldsymbol{\beta}\\
&\qquad - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) + \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\boldsymbol{\beta}\\
&= \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) + \boldsymbol{\lambda}^T\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) - \boldsymbol{\beta}\right)\\
&\qquad - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) - \boldsymbol{\beta}\right)\\
&= \underbrace{\boldsymbol{\lambda}^T\boldsymbol{\beta} + \underbrace{\boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})}_{\mathbf{a}^T}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})}_{\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})} + \underbrace{(\boldsymbol{\lambda}^T - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})}_{\mathbf{d}^T}\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta},\mathbf{y}) - \boldsymbol{\beta}\right)\\
&= \hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) + \mathbf{d}^T(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta})
\end{aligned} \tag{2.18}$$

dengan

$$\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) = \boldsymbol{\lambda}^T\boldsymbol{\beta} + \mathbf{a}^T(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})$$

$$\mathbf{a}^T = \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})$$

$$\mathbf{d}^T = \boldsymbol{\lambda}^T - \boldsymbol{\omega}^T\mathbf{GZ}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X} = \boldsymbol{\lambda}^T - \mathbf{a}^T\mathbf{X}$$

Kemudian didefinisikan $\boldsymbol{t} = \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v}$, sehingga diperoleh MSE dari $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta},\mathbf{y})$ berikut:

$$\begin{aligned}
\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta},\mathbf{y})] &= E[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta},\mathbf{y}) - \boldsymbol{t}]^2\\
&= E[\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) + \mathbf{d}^T\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right) - \boldsymbol{t}]^2\\
&= E[(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t}) + \mathbf{d}^T\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)]^2\\
&= E[\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t}]^2 + E\left[\mathbf{d}^T\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)\right]^2\\
&\quad +2cov[(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t}), \mathbf{d}^T\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)]
\end{aligned}$$

karena $(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t})$ dan $\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)$ independen (dibuktikan pada Lampiran 4), maka:

$$\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta},\mathbf{y})] = E[\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t}]^2 + E[\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)]^2$$
$$= g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta})$$

dengan

$$\begin{aligned}
g_1(\boldsymbol{\delta}) &= E[\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t}]^2\\
&= E[(\boldsymbol{\lambda}^T\boldsymbol{\beta}+\mathbf{a}^{\boldsymbol{T}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\\
&\quad -(\boldsymbol{\lambda}^T\boldsymbol{\beta}+\boldsymbol{\omega}^T\mathbf{v}))(\boldsymbol{\lambda}^T\boldsymbol{\beta}+(\mathbf{a}^{\boldsymbol{T}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})-(\boldsymbol{\lambda}^T\boldsymbol{\beta}+\boldsymbol{\omega}^T\mathbf{v}))^T]\\
&= E[((\mathbf{a}^{\boldsymbol{T}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})-\boldsymbol{\omega}^T\mathbf{v})((\mathbf{a}^{\boldsymbol{T}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})-\boldsymbol{\omega}^T\mathbf{v})^T]\\
&= E[((\mathbf{a}^{\boldsymbol{T}}(\mathbf{Zv}+\mathbf{e})-\boldsymbol{\omega}^T\mathbf{v})((\mathbf{Zv}+\mathbf{e})^{\boldsymbol{T}}\mathbf{a}-\mathbf{v}^T\boldsymbol{\omega})]\\
&= E[(\mathbf{a}^{\boldsymbol{T}}(\mathbf{Zv}+\mathbf{e})-\boldsymbol{\omega}^T\mathbf{v})(\mathbf{Zv}+\mathbf{e})^{\boldsymbol{T}}\mathbf{a}-(\mathbf{a}^{\boldsymbol{T}}(\mathbf{Zv}+\mathbf{e})-\boldsymbol{\omega}^T\mathbf{v})\mathbf{v}^T\boldsymbol{\omega}]\\
&= E[(\mathbf{a}^{\boldsymbol{T}}(\mathbf{Zv}+\mathbf{e})-\boldsymbol{\omega}^T\mathbf{v})(\mathbf{Zv}+\mathbf{e})^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{ZG}^T\boldsymbol{\omega}]\\
&\quad -E[(\mathbf{a}^{\boldsymbol{T}}(\mathbf{Zv}+\mathbf{e})-\boldsymbol{\omega}^T\mathbf{v})\mathbf{v}^T\boldsymbol{\omega}]\\
&= 0-E[(\mathbf{a}^{\boldsymbol{T}}(\mathbf{Zv}+\mathbf{e})-\boldsymbol{\omega}^T\mathbf{v})\mathbf{v}^T\boldsymbol{\omega}]\\
&= E[(\boldsymbol{\omega}^T\mathbf{v}-\mathbf{a}^{\boldsymbol{T}}(\mathbf{Zv}+\mathbf{e}))\mathbf{v}^T\boldsymbol{\omega}]\\
&= E[(\boldsymbol{\omega}^T\mathbf{v}-\boldsymbol{\omega}^T\mathbf{GZ}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}(\delta)(\mathbf{Zv}+\mathbf{e}))\mathbf{v}^T\boldsymbol{\omega}]\\
&= E[\boldsymbol{\omega}^T(\mathbf{v}-\mathbf{GZ}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}(\delta)(\mathbf{Zv}+\mathbf{e}))\mathbf{v}^T\boldsymbol{\omega}]\\
&= \boldsymbol{\omega}^T E[\mathbf{vv}^T-\mathbf{GZ}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}(\delta)(\mathbf{Zvv}^T+\mathbf{ev}^T)]\boldsymbol{\omega}\\
&= \boldsymbol{\omega}^T(\mathbf{G}-\mathbf{GZ}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{ZG})\boldsymbol{\omega}.
\end{aligned}$$

$$\begin{aligned}
g_2(\boldsymbol{\delta}) &= E[\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\delta)-\boldsymbol{\beta}\big)]^2\\
&= E[\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)(\mathbf{d}^T\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big))^T]\\
&= E\left[\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)\big(\widehat{\boldsymbol{\beta}}(\delta)-\boldsymbol{\beta}\big)^{\boldsymbol{T}}\mathbf{d}\right]\\
&= E\left[\begin{matrix}\mathbf{d}^{\boldsymbol{T}}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\\(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}\end{matrix}\right]\\
&= \mathbf{d}^{\boldsymbol{T}}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\\
&\quad E[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^{\boldsymbol{T}}]\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}\\
&= \mathbf{d}^{\boldsymbol{T}}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{V}(\boldsymbol{\delta})\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}\\
&= \mathbf{d}^{\boldsymbol{T}}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{IX}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}\\
&= \mathbf{d}^{\boldsymbol{T}}((\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}\\
&= \mathbf{d}^{\boldsymbol{T}}\mathbf{I}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}\\
&= \mathbf{d}^{\boldsymbol{T}}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}.
\end{aligned}$$

Dengan demikian, bentuk MSE BLUP yang diperoleh, sebagai berikut:

$$\text{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}, \mathbf{y})] = E[\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{y}) - \boldsymbol{t}]^2 + E[\mathbf{d}^{\boldsymbol{T}}(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta})]^2$$

$$= g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) \quad (2.19)$$

dimana

$$g_1(\boldsymbol{\delta}) = \boldsymbol{\omega}^T(\mathbf{G} - \mathbf{G}\mathbf{Z}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}\mathbf{Z}\mathbf{G})\boldsymbol{\omega}$$

$$g_2(\boldsymbol{\delta}) = \mathbf{d}^T(\mathbf{X}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}\mathbf{X})^{-1}\mathbf{d}.$$

2.9 Metode EBLUP pada GLMM dan Taksiran MSE EBLUP

Seperti yang telah dijelaskan sebelumnya, pada metode BLUP, diasumsikan bahwa $\boldsymbol{\delta}$ diketahui nilainya. Dengan menggunakan metode BLUP, didapat $\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}, \mathrm{y})$ dan $\hat{\boldsymbol{v}}(\boldsymbol{\delta}, \mathbf{y})$ sebagai berikut:

$$\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}, \mathbf{y}) = (\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathrm{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{y}$$

dan

$$\hat{\boldsymbol{v}}(\boldsymbol{\delta}, \mathbf{y}) = \mathbf{G}(\boldsymbol{\delta})\mathrm{Z}^{\boldsymbol{T}}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathrm{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}, \mathbf{y}))$$

Dengan demikian, taksiran BLUP untuk $\boldsymbol{\tau} = \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\boldsymbol{v}$ adalah:

$$\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}, \mathbf{y}) = \boldsymbol{\lambda}^T\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}, \mathbf{y}) + \boldsymbol{\omega}^T\,\hat{\boldsymbol{v}}(\boldsymbol{\delta}, \mathbf{y})$$

yang memiliki MSE minimum dari semua taksiran *unbiased* dan linear.

Namun pada kenyataannya, nilai $\boldsymbol{\delta}$ tidak diketahui, sehingga perlu ditaksir dari data. Salah satu metode yang dapat digunakan adalah metode *Maximum Likelihood* (ML). Dengan metode ML, diperlukan fungsi *likelihood* yang merupakan pdf bersama dari $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ dimana **v** dan **e** diasumsikan berdistribusi normal dengan vektor mean dan matriks varians-kovarians masing-masing yaitu $(\mathbf{0}, \mathbf{G})$ dan $(\mathbf{0}, \mathbf{R})$. Karena **v** dan **e** berdistribusi normal, maka **y** juga berdistribusi normal dan berdasarkan persamaan (2.5) dan (2.6), dapat dituliskan sebagai $\mathbf{y} \sim \mathrm{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{V}(\boldsymbol{\delta}))$. Oleh karena itu, diperoleh fungsi *likelihood* sebagai berikut:

$$L(\boldsymbol{\beta}, \boldsymbol{\delta}; \boldsymbol{y}) = \frac{1}{(2\pi)^{n/2}|\mathbf{V}(\boldsymbol{\delta})|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right] \quad (2.20)$$

Dari persamaan (2.20) tersebut, kemudian diperoleh fungsi *log-likelihood* berikut yang akan digunakan dalam metode ML:

$$\ln L(\boldsymbol{\beta},\boldsymbol{\delta};\boldsymbol{y}) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln(|\mathbf{V}(\boldsymbol{\delta})|) - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})$$
$$= c - \frac{1}{2}(\ln(|\mathbf{V}(\boldsymbol{\delta})|) + (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}))$$

dengan c suatu konstanta.

Selanjutnya, dengan menggunakan differensiasi parsial pada matriks, fungsi *log-likelihood* tersebut diturunkan terhadap $\boldsymbol{\delta}$ dan dinotasikan dengan $\mathbf{s}(\boldsymbol{\beta},\boldsymbol{\delta})$. Untuk elemen ke-$j$ dari $\mathbf{s}(\boldsymbol{\beta},\boldsymbol{\delta})$dapat dituliskan sebagai berikut:

$$\mathrm{s}_j(\boldsymbol{\beta},\boldsymbol{\delta}) = \frac{\partial \ln L(\boldsymbol{\beta},\boldsymbol{\delta};\boldsymbol{y})}{\partial \delta_j}$$
$$= \frac{\partial\left(c - \frac{1}{2}\left(\ln(|\mathbf{V}(\boldsymbol{\delta})|) + (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)\right)}{\partial \delta_j}$$
$$= \frac{\partial c}{\partial \delta_j} - \frac{1}{2}\frac{\partial \ln(|\mathbf{V}(\boldsymbol{\delta})|)}{\partial \delta_j} - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T\frac{\partial \mathbf{V}^{-1}(\boldsymbol{\delta})}{\partial \delta_j}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})$$
$$= -\frac{1}{2}tr\left(\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{V}_{(j)}\right) + \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T\left(\mathbf{V}^{-1}\mathbf{V}_{(j)}\mathbf{V}^{-1}\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \quad (2.21)$$

dengan $\mathbf{V}_{(j)} = \frac{\partial \mathbf{V}(\boldsymbol{\delta})}{\partial \delta_j}$.

Sesuai dengan prosedur metode ML, selanjutnya akan dicari solusi dari persamaan $\mathrm{s}_j(\boldsymbol{\beta},\boldsymbol{\delta}) = 0$, sehingga diperoleh:

$$tr\left(\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{V}_{(j)}\right) = \left(\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{V}_{(j)}\mathbf{V}^{-1}(\boldsymbol{\delta})\right)\left(\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}}\right) \quad (2.22)$$

Namun, dari persamaan (2.22), terlihat bahwa $\boldsymbol{\delta}$ tidak dapat dicari solusinya. Oleh karena itu, taksiran $\boldsymbol{\delta}$ dengan metode ML, diselesaikan secara iteratif dengan menggunakan *Scoring Algorithm*. Sebelumnya akan dijelaskan terlebih dahulu mengenai *Scoring Algorithm*.

Scoring Algorithm merupakan salah satu metode numerik dari Metode Newton-Raphson. Berikut adalah bentuk formula metode Newton-Raphson:

$$\widehat{\boldsymbol{\delta}}^{(a+1)} = \widehat{\boldsymbol{\delta}}^{(a)} - \left(\mathbf{H}\left(\widehat{\boldsymbol{\delta}}^{(a)}\right)\right)^{-1}\mathbf{s}\left(\widehat{\boldsymbol{\delta}}^{(a)}\right) \quad a = 1, 2, \ldots$$

dengan

$\mathbf{s}\left(\widehat{\boldsymbol{\delta}}^{(a)}\right)$ adalah vektor turunan parsial pertama dari $l(\boldsymbol{\delta})$, pada $\boldsymbol{\delta} = \widehat{\boldsymbol{\delta}}^{(a)}$. $l(\boldsymbol{\delta})$ merupakan fungsi *log-likelihood* dari vektor $\boldsymbol{\delta}$.

$\mathbf{H}\left(\widehat{\boldsymbol{\delta}}^{(a)}\right)$ adalah matriks turunan parsial kedua dari $l(\boldsymbol{\delta})$, pada $\boldsymbol{\delta} = \widehat{\boldsymbol{\delta}}^{(a)}$.

Dengan mengganti $\mathbf{H}(\hat{\boldsymbol{\delta}}^{(a)})$ dengan nilai ekspektasinya, $E\left[\mathbf{H}(\hat{\boldsymbol{\delta}}^{(a)})\right]$, maka akan diperoleh *Scoring Algorithm* yang akan digunakan untuk mencari penyelesaian dari persamaan (2.22) sebagai berikut:

$$\hat{\boldsymbol{\delta}}^{(a+1)} = \hat{\boldsymbol{\delta}}^{(a)} + (\boldsymbol{\Upsilon}(\hat{\boldsymbol{\delta}}^{(a)}))^{-\mathbf{1}}\mathbf{s}\left(\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}^{(a)}), \hat{\boldsymbol{\delta}}^{(a)}\right)$$

dengan

$$\mathbf{s}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\delta}}) = (\mathrm{s}_1(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\delta}}), \mathrm{s}_2(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\delta}}), \dots, \mathrm{s}_q(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\delta}}))^T$$

$$\boldsymbol{\Upsilon}(\hat{\boldsymbol{\delta}}) = (\boldsymbol{\Upsilon}_1^T(\hat{\boldsymbol{\delta}}), \boldsymbol{\Upsilon}_2^T(\hat{\boldsymbol{\delta}}), \dots, \boldsymbol{\Upsilon}_q^T(\hat{\boldsymbol{\delta}}))^T$$

dimana

$$\boldsymbol{\Upsilon}_{jk}(\boldsymbol{\delta}) = -E\left(\frac{\partial^2 \ln L(\boldsymbol{\beta},\delta;\boldsymbol{y})}{\partial\delta_k \partial\delta_j}\right) = \frac{1}{2} tr(\mathbf{V}^{-\mathbf{1}}(\hat{\boldsymbol{\delta}})\mathbf{V}_{(j)}\mathbf{V}^{-\mathbf{1}}(\hat{\boldsymbol{\delta}})\mathbf{V}_{(k)}) \qquad (2.23)$$

(dibuktikan pada Lampiran 5).

Proses iterasi berhenti jika $\hat{\boldsymbol{\delta}}^{(a+1)} \approx \hat{\boldsymbol{\delta}}^{(a)}$, misalkan dengan $\left|\hat{\boldsymbol{\delta}}^{(a+1)} - \hat{\boldsymbol{\delta}}^{(a)}\right| < 10^{-5}$. Kemudian $\hat{\boldsymbol{\delta}}^{(a+1)}$ dapat diambil sebagai taksiran dari $\boldsymbol{\delta}$.

Setelah diperoleh taksiran untuk $\boldsymbol{\delta}$, yang dinotasikan dengan $\hat{\boldsymbol{\delta}}(\mathbf{y})$, maka akan didapat taksiran parameter baru, dengan mensubstitusikan $\hat{\boldsymbol{\delta}}(\mathbf{y})$ ke $\hat{\boldsymbol{\beta}}$ dan $\hat{\boldsymbol{\nu}}$ pada persamaan (2.16) yaitu:

$$\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}(\mathbf{y}), \mathbf{y}) = (\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\hat{\boldsymbol{\delta}})\mathbf{y}$$

dan (2.24)

$$\hat{\boldsymbol{\nu}}(\hat{\boldsymbol{\delta}}(\mathbf{y}), \mathbf{y}) = \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\hat{\boldsymbol{\delta}})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}(\mathbf{y}), \mathbf{y}))$$

Berdasarkan persamaan (2.24), maka taksiran $\boldsymbol{\tau} = \boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\boldsymbol{\nu}$ baru yang disebut dengan taksiran EBLUP adalah sebagai berikut:

$$\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}(\mathrm{y}), \mathbf{y}) = \boldsymbol{\lambda}^{\boldsymbol{T}}\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}(\mathrm{y}), \mathrm{y}) + \boldsymbol{\omega}^{\boldsymbol{T}}\hat{\boldsymbol{\nu}}(\hat{\boldsymbol{\delta}}(\mathrm{y}), \mathrm{y}). \qquad (2.25)$$

Untuk mengetahui seberapa baik taksiran EBLUP yang telah didapat tersebut, akan dicari MSE EBLUP nya, yaitu:

$$\begin{aligned}\mathrm{MSE}\left[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}})\right] &= E[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \boldsymbol{t}]^2 \\ &= E[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \boldsymbol{t} + \hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^2 \\ &= E[(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}) + (\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta}))]^2\end{aligned}$$

$$= \underbrace{E[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}]^2}_{\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]} + E[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^2 + 2cov[(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}), (\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta}))]$$

$$= \mathrm{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})] + E[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^2 + 2cov[(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}), (\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta}))]$$

Karena $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ dan $\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})$ independen (dibuktikan pada Lampiran 6), maka diperoleh:

$$\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}})] = \underbrace{\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]}_{i} + \underbrace{E[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^{\mathbf{2}}}_{ii} \qquad (2.26)$$

dimana

$\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]$ merupakan bentuk dari MSE BLUP.

Karena bagian (ii) pada persamaan (2.26) nilainya sangat sulit dicari, maka digunakan aproksimasi Taylor untuk menghitungnya. Kemudian didapat:

$$E[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^{\mathbf{2}} \approx tr\left[\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \delta}\right) \mathbf{V} \left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \delta}\right)^T I_\delta^{-1}(\boldsymbol{\delta})\right] =: g_3(\boldsymbol{\delta}) \qquad (2.27)$$

dimana

$$I_\delta^{-1}(\boldsymbol{\delta}) = var(\hat{\boldsymbol{\delta}}) = \frac{1}{2} tr(\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}}) \frac{\partial \mathbf{V}(\boldsymbol{\delta})}{\partial \delta_j} \mathbf{V}^{-\mathbf{1}}(\hat{\boldsymbol{\delta}}) \frac{\partial \mathbf{V}(\boldsymbol{\delta})}{\partial \delta_k})$$

(dibuktikan pada Lampiran 7).

Berdasarkan persamaan (2.19) dan (2.27), didapatkan aproksimasi kedua untuk $\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]$ pada persamaan (2.26) sebagai berikut:

$$\mathrm{MSE}[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}})] \approx g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta}) \qquad (2.28)$$

dengan

$$g_1(\boldsymbol{\delta}) = \boldsymbol{\omega}^T(\mathbf{G} - \mathbf{G}\mathbf{Z}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}\mathbf{Z}\mathbf{G})\boldsymbol{\omega}$$

$$g_2(\boldsymbol{\delta}) = \mathbf{d}^T(\mathbf{X}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}\mathbf{X})^{-1}\mathbf{d}$$

$$g_3(\boldsymbol{\delta}) \approx tr\left[\left(\frac{\partial \mathbf{a}^{T}}{\partial \delta}\right) \mathbf{V} \left(\frac{\partial \mathbf{a}^{T}}{\partial \delta}\right)^T I_\delta^{-1}(\boldsymbol{\delta})\right].$$

Pada metode EBLUP, diasumsikan bahwa $\boldsymbol{\delta}$ tidak diketahui. Dengan demikian, MSE EBLUP yang diperoleh pada persamaan (2.28) juga masih bergantung pada nilai $\boldsymbol{\delta}$ yang tidak diketahui, sehingga nilai MSE EBLUP tidak dapat dicari. Oleh karena itu, perlu dilakukan penaksiran terhadap nilai MSE EBLUP tersebut.

Ada beberapa cara untuk menaksir MSE EBLUP. Cara pertama, dengan mengganti $\boldsymbol{\delta}$ dengan $\hat{\boldsymbol{\delta}}$ pada MSE BLUP di persamaan (2.19), sehingga diperoleh:

$$\mathrm{mse}_{\boldsymbol{N}}[\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}})] = g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}) \qquad (2.29)$$

dan dikenal dengan *naive estimator*. Akan tetapi, *naive estimator* ini tidak mengikutsertakan penambahan variasi $\boldsymbol{\delta}$ yaitu bagian (ii) dari persamaan (2.26), sehingga tidak sesuai dengan MSE EBLUP. Kemudian digunakan cara lain untuk menaksir MSE EBLUP, yaitu dengan mensubstitusikan $\widehat{\boldsymbol{\delta}}$ ke $\boldsymbol{\delta}$ pada aproksimasi kedua MSE EBLUP di persamaan (2.28). Hasil taksiran MSE EBLUP yang didapat yaitu:

$$\text{mse}\left[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})\right] = g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + g_3(\widehat{\boldsymbol{\delta}}). \tag{2.30}$$

Namun ternyata taksiran MSE EBLUP pada persamaan (2.30) bias terhadap $\text{MSE}\left[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})\right]$ (dibuktikan pada Lampiran 8). Oleh karena itu, perlu dicari taksiran dari aproksimasi MSE EBLUP yang *unbiased* untuk $\text{MSE}\left[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})\right]$ dan diperoleh:

$$\begin{aligned}
E\left\{\text{mse}\left[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})\right]\right\} &= E\{g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + g_3(\widehat{\boldsymbol{\delta}})\} \\
&= E[g_1(\widehat{\boldsymbol{\delta}})] + E[g_2(\widehat{\boldsymbol{\delta}})] + E[g_3(\widehat{\boldsymbol{\delta}})] \\
&\approx g_1(\boldsymbol{\delta}) + c_{\boldsymbol{\delta}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta})
\end{aligned}$$

$$E\{g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + g_3(\widehat{\boldsymbol{\delta}})\} \approx g_1(\boldsymbol{\delta}) + c_{\widehat{\boldsymbol{\delta}}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta})$$

$$E\{g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + g_3(\widehat{\boldsymbol{\delta}})\} - c_{\widehat{\boldsymbol{\delta}}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta}) \approx g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta})$$

$$E\{g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + g_3(\widehat{\boldsymbol{\delta}}) - c_{\widehat{\boldsymbol{\delta}}}^T(\widehat{\boldsymbol{\delta}})\nabla g_1(\widehat{\boldsymbol{\delta}}) + g_3(\widehat{\boldsymbol{\delta}})\} \approx g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta})$$

$$E\{g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + 2g_3(\widehat{\boldsymbol{\delta}}) - c_{\widehat{\boldsymbol{\delta}}}^T(\widehat{\boldsymbol{\delta}})\nabla g_1(\widehat{\boldsymbol{\delta}})\} \approx g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta})$$

$$E\{g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + 2g_3(\widehat{\boldsymbol{\delta}}) - c_{\widehat{\boldsymbol{\delta}}}^T(\widehat{\boldsymbol{\delta}})\nabla g_1(\widehat{\boldsymbol{\delta}})\} \approx \text{MSE}[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})]$$

Dengan demikian, taksiran yang *unbiased* untuk $\text{MSE}[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})]$ adalah:

$$\text{mse}\left[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})\right] = g_1(\widehat{\boldsymbol{\delta}}) + g_2(\widehat{\boldsymbol{\delta}}) + 2g_3(\widehat{\boldsymbol{\delta}}) - c_{\boldsymbol{\delta}}^T(\widehat{\boldsymbol{\delta}})\nabla g_1(\widehat{\boldsymbol{\delta}}) \tag{2.31}$$

BAB 3
METODE *EMPIRICAL BEST LINEAR UNBIASED PREDICTION* (EBLUP) PADA MODEL FAY-HERRIOT

Pada bab 3 ini, akan dibahas mengenai model *small area*, termasuk model area level, serta model Fay-Herriot yang merupakan kasus dari model area level. Kemudian akan dicari taksiran parameter pada model Fay-Harriot. Metode yang akan digunakan adalah metode *Empirical best Linear Unbiased Prediction* (EBLUP). Setelah diperoleh taksiran parameter EBLUP, pembahasan kemudian dilanjutkan dengan mencari MSE EBLUP dan taksiran MSE EBLUP.

3.1 Model *Small Area*

Small area sering digunakan sebagai gambaran sebuah area geografis kecil, seperti propinsi, kabupaten, kecamatan, maupun kelurahan/desa dari suatu negara. Selain itu, *small area* dapat juga menggambarkan suatu subpopulasi kecil untuk demografi maupun kelompok orang yang memiliki sosial ekonomi (ras, jenis kelamin, umur) tertentu yang berada dalam area geografis yang lebih luas (Rao, 2003).

Penaksiran parameter di *small area* dapat dilakukan dengan dua cara, yaitu penaksiran langsung (*direct estimation*) dan penaksiran tidak langsung (*indirect estimation*). Model desain penarikan sampel digunakan pada penaksiran langsung. Akan tetapi, penaksiran ini tidak dapat menghasilkan ketelitian yang cukup (*standar error* nya besar) jika ukuran sampel dari *small area* yang diamati kecil, sehingga diperlukan motode penaksiran tidak langsung. Suatu teknik statistika untuk menaksir parameter pada *small area* yang ukuran sampelnya kecil, dengan memanfaatkan informasi tambahan (*auxiliary variable*), seperti data sensus dan atau catatan administratif *small area* tersebut, maupun catatan administratif *small area* lain yang hampir sama, dinamakan *small area estimation* (Rao, 2003).

Seperti yang telah dijelaskan sebelumnya, pada subbab 2.7, bahwa pada model regresi linear berganda, pengaruh tetap yang terkandung di dalam model telah diperhatikan, yaitu **X**. Akan tetapi, pengaruh acak, yaitu **v**, tidak terkandung di dalam model. Sedangkan pada *general linear mixed model* (GLMM) yang telah dituliskan dalam persamaan (2.2), pengaruh tetap dan pengaruh acak telah terkandung di dalam model.

Salah satu model yang merupakan kasus khusus dari GLMM adalah model *small area*. Karena pengaruh tetap dan pengaruh acak *small area* terkandung dalam model ini, variasi antar *small area* yang tidak dijelaskan oleh pengaruh tetap, dijelaskan melalui pengaruh acak tersebut. Berdasarkan ketersediaan datanya, model *small area* dapat dikelompokkan menjadi dua yaitu model area level dan model unit level. Namun, dalam tugas akhir ini yang akan dijelaskan lebih lanjut hanya model area level.

3.1.1 Model Area Level

Model area level adalah salah satu jenis model *small area*, dimana data pendukung yang tersedia hanya sampai pada tingkat area, yaitu $\mathbf{z}_i^T = (z_{1i}, \dots, z_{pi})$. Parameter *small area* yang ingin diamati adalah θ_i. Parameter *small area* ini berhubungan linear dengan $\mathbf{z}_i^T$ mengikuti model linear berikut:

$$\theta_i = \mathbf{z}_i^T \boldsymbol{\beta} + b_i v_i \quad i = 1, \dots, m \tag{3.1}$$

dengan

$\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^T$ adalah vektor parameter yang *fixed*, berukuran $px1$

b_i konstanta positif yang diketahui

v_i pengaruh acak *small area*, diasumsikan $v_i \sim iid(0, \sigma_v^2)$

m jumlah *small area*.

Namun, dalam membuat kesimpulan tentang populasi di bawah model (3.1), diasumsikan bahwa penaksir langsung $\hat{\theta}_i$ telah ada pada model dan dapat dituliskan sebagai:

$$\hat{\theta}_i = \theta_i + e_i \quad i = 1, \dots, m \tag{3.2}$$

dimana e_i adalah sampling *error* yang diasumsikan diketahui dengan $e_i \sim ind(0, \psi_i)$.

Model pada persamaan (3.1) dan (3.2), jika digabungkan akan menjadi:

$$\hat{\theta}_i = \mathbf{z}_i^T \boldsymbol{\beta} + b_i v_i + e_i \quad i = 1, \dots, m \tag{3.3}$$

dengan asumsi bahwa $v_i \sim iid(0, \sigma_v^2)$ dan $e_i \sim ind(0, \psi_i)$. Model ini merupakan salah satu kasus khusus dari *general linear mixed model* (GLMM) dengan *block diagonal covariance structure*, yaitu GLMM dengan matriks varians-kovarians dari v_i, e_i, dan $\hat{\theta}_i$ masing-masing sebagai berikut:

$$\mathbf{G} = \begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_n \end{pmatrix}$$

dan (3.4)

$$\mathbf{V} = \mathbf{G} + \mathbf{R} = \begin{pmatrix} \sigma_v^2 + \psi_1 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 + \psi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 + \psi_n \end{pmatrix}$$

Pada model area level ini, yang akan ditaksir adalah parameter *small area*, yaitu $\theta_i = \mathbf{z}_i^T \boldsymbol{\beta} + b_i v_i \quad i = 1, \dots, m$.

3.2 Model Fay-Herriot

Model ini diperkenalkan oleh Fay dan Herriot (1979) sebagai model dasar untuk menaksir pendapatan per kapita pada *small area* - *small area* (dengan populasi yang kurang dari 1.000 jiwa penduduk) di Amerika Serikat. Model Fay-Herriot ini merupakan kasus model area level seperti pada persamaan (3.3) dengan $b_i = 1$.

Berikut ini akan didefinisikan bentuk model Fay-Harriot yang memiliki nilai taksiran langsung $\hat{\theta}_i$ dan vektor variabel pendukung $\mathbf{z}_\mathbf{i}^\mathbf{T}$. Untuk selanjutnya, model ini akan digunakan dalam penaksiran parameter dan perhitungan nilai MSE dengan metode EBLUP.

Bentuk model Fay-Herriot (Rao, 2003) adalah sebagai berikut:

$$\begin{aligned} \hat{\theta}_i &= \mathbf{z}_\mathbf{i}^\mathbf{T} \boldsymbol{\beta} + v_i + e_i \qquad i = 1, 2, \dots, m \\ &= \theta_i + e_i \end{aligned} \tag{3.5}$$

dengan

$\hat{\theta}_i$: nilai taksiran langsung, berukuran $1x1$

θ_i : parameter *small area*, berukuran $1x1$

$\mathbf{z}_i$: vektor variabel pendukung, berukuran $px1$

$\boldsymbol{\beta}$: vektor parameter yang *fixed*, berukuran $px1$

v_i : pengaruh acak *small area*, berukuran $1x1$, diasumsikan $v_i \sim$ NID $(0, \sigma_v^2)$

e_i : sampling *error*, berukuran $1x1$, diasumsikan $e_i \sim$ NID $(0, \psi_i)$, ψ_i diketahui,

dimana v_i dan e_i saling independen, sehingga $E(v_i e_i^T) = E(e_i v_i^T) = 0$ dan $E(v_i v_i^T) = \sigma_v^2$. Matriks varians-kovarians dari v_i dan e_i yaitu masing-masing **G** dan **R** merupakan matriks *block* diagonal dan memiliki bentuk sebagai berikut:

$$\mathbf{G} = \begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix}$$

dan (3.6)

$$\mathbf{R} = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_n \end{pmatrix}$$

Matriks **G** merupakan matriks varians-kovarians, definit positif, berukuran nxn, dari variansi antar *small area*, yang biasanya tidak diketahui dan harus ditaksir. Matriks **R** adalah matriks varians-kovarians definit positif berukuran nxn dari *sampling error*. Sedangkan $\mathbf{V} = \mathbf{G} + \mathbf{R}$ adalah matriks varians-kovarians dari $\hat{\theta}_i$, yang berbentuk:

$$\mathbf{V} = \begin{pmatrix} \sigma_v^2 + \psi_1 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 + \psi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 + \psi_n \end{pmatrix} \qquad (3.7)$$

Pada model Fay-Herriot ini, parameter yang akan ditaksir adalah $\theta_i = \mathbf{z}_i^{\mathbf{T}} \boldsymbol{\beta} + v_i$. Metode yang akan digunakan adalah metode EBLUP.

3.3 Penaksir Langsung, Sintetik, dan Komposit pada Model Fay-Herriot

3.3.1 Penaksir Langsung (*direct estimator*)

Pada model Fay-Herriot yang telah dijelaskan sebelumnya, parameter *small area* (θ_i) ditaksir secara langsung dengan $\hat{\theta}_i$. Nilai $\hat{\theta}_i$ ini berhubungan secara linear dengan θ_i mengikuti model linear berikut:

$$\hat{\theta}_i = \underbrace{\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta} + v_i}_{\theta_i} + e_i = \theta_i + e_i \quad i = 1, \dots, m \tag{3.8}$$

dimana e_i adalah sampling *error* pada area ke$-i$ yang diasumsikan diketahui dengan $e_i \sim \text{NID}\ (0, \psi_i)$.

Penaksiran langsung ini dilakukan hanya berdasarkan pada data dari sampel dalam *small area*. Akan tetapi, taksiran langsung yang dihasilkan memiliki *standar error* yang besar karena ukuran sampel dari *small area* yang diamati terlalu kecil. Oleh karena itu, diperlukan penaksiran tidak langsung (*indirect estimation*) yang dapat memperkecil *standar error* (Rao, 2003).

3.3.2 Penaksir Sintetik

Penaksir sintetik dihasilkan dari metode sintetik, yang merupakan salah satu metode penaksiran tidak langsung dalam *small area estimation.* Penaksiran tidak langsung dilakukan untuk meningkatkan keakuratan dalam penaksiran (memperkecil *standar error*) dengan memanfaatkan informasi dari *small area* lain yang memiliki karakteristik hampir sama.

Penaksir sintetik dengan informasi variabel-variabel pendukung (*auxiliary variables*) yang tersedia pada area ke$-i$, yaitu $\mathbf{z}_\mathbf{i}^\mathbf{T}$, dapat dinyatakan dalam bentuk:

$$\hat{\theta}_i^{\ S} = \mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}\ ;\ i = 1, 2, \dots, m \tag{3.9}$$

dimana

$\mathbf{z}_\mathbf{i}$ berukuran $px1$,

$\hat{\boldsymbol{\beta}}$ merupakan vektor parameter yang *fixed,* berukuran $px1$.

Definisi penaksir sintetik telah dijelaskan oleh Gonzales (1973) dalam Rao (2003), bahwa suatu penaksir disebut penaksir sintetik jika suatu penaksir langsung yang *reliable* untuk area luas yang mencakup *small area - small area*, digunakan untuk menurunkan penaksir tidak langsung untuk *small area*, di bawah asumsi bahwa *small area* tersebut memiliki karakteristik yang sama seperti area yang luas. Namun, kelemahan dari penaksir sintetik ini adalah memiliki bias yang besar. Hal ini dikarenakan adanya asumsi kesamaan karakteristik antara *small area* dengan area yang luas.

3.3.3 Penaksir Komposit

Penaksiran pada *small area* dengan menggunakan dua jenis penaksir (penaksir langsung dan penaksir sintetik) yang telah dijelaskan sebelumnya, memiliki kelebihan dan kekurangan masing-masing. Penaksir langsung bersifat *unbiased* karena hanya berdasarkan pada data sampel dari *small area* tersebut. Namun, penaksir ini kurang stabil, karena memiliki *standar error* yang besar. Sedangkan penaksir sintetik telah memiliki *standar error* yang lebih baik daripada penaksir langsung, tetapi memiliki bias yang besar. Oleh karena itu, salah satu cara untuk menyeimbangkan ketidakstabilan dari penaksir langsung dan bias yang besar dari penaksir sintetik, dilakukan dengan membentuk penaksir komposit.

Penaksir komposit merupakan rata-rata terboboti dari penaksir langsung dan penaksir sintetik. Bentuk penaksir komposit yaitu:

$$\begin{aligned} \hat{\theta}_i^{\,C} &= \phi_i \hat{\theta}_i + (1-\phi_i)\hat{\theta}_i^{\,S}\ ;\ i = 1, 2, \dots, m \\ &= \phi_i \hat{\theta}_i + (1-\phi_i)\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}} \end{aligned} \tag{3.10}$$

dengan ϕ_i merupakan bobot yang dipilih, dimana $0 \leq \phi_i \leq 1$ dan m menunjukkan banyaknya *small area*.

Bobot optimal untuk penaksir komposit dapat diperoleh dengan meminimalkan MSE $\left[\hat{\theta}_i^{\,C}\right]$ terhadap ϕ_i dengan asumsi bahwa $cov\left(\hat{\theta}_i, \hat{\theta}_i^{\,S}\right) = 0$. Bobot optimalnya dapat dinyatakan sebagai:

$$\phi_i^{\,*} = \frac{\mathrm{MSE}\left[\hat{\theta}_i^{\,S}\right]}{\mathrm{MSE}\left[\hat{\theta}_i\right] + \mathrm{MSE}\left[\hat{\theta}_i^{\,S}\right]} \tag{3.11}$$

(Rao, 2003).

3.4 Penaksir BLUP pada Model Fay-Herriot

Berdasarkan penaksir komposit pada persamaan (3.10), dapat dibentuk penaksir pada model Fay-Herriot yang bersifat linear, *unbiased*, dan memiliki *standar error* kecil (MSE terbaik), yang berbentuk (Rao, 2003) :

$$\begin{aligned}\left(T(\hat{\theta}_i)\right)(\sigma_v^2) &= \phi_i\hat{\theta}_i + (1-\phi_i)\mathbf{z_i^T}\hat{\boldsymbol{\beta}} \\ &= \phi_i\hat{\theta}_i - \phi_i\mathbf{z_i^T}\hat{\boldsymbol{\beta}} + \mathbf{z_i^T}\hat{\boldsymbol{\beta}} \\ &= \mathbf{z_i^T}\hat{\boldsymbol{\beta}} + \phi_i(\hat{\theta}_i - \mathbf{z_i^T}\hat{\boldsymbol{\beta}})\end{aligned} \tag{3.12}$$

dengan

$$\phi_i = \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}$$

$$\hat{\boldsymbol{\beta}}(\sigma_v^2) = \left(\boldsymbol{z_i}\left(\sigma_v^2 + \psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\boldsymbol{z_i}\left(\sigma_v^2 + \psi_i\right)^{-1}\hat{\theta}_i$$

Penaksir BLUP yang didapat, $T(\hat{\theta}_i)$ masih mengandung σ_v^2, karena pada metode BLUP diasumsikan bahwa σ_v^2 diketahui nilainya. $\hat{\boldsymbol{\beta}}$ yang diperoleh tersebut, dibuktikan pada Lampiran 9.

3.5 MSE BLUP pada Model Fay-Herriot

Setelah diperoleh nilai taksiran BLUP pada subbab 3.4, kemudian perlu dicari nilai MSE BLUP untuk mengukur seberapa baik taksiran BLUP tersebut. Untuk mempermudah dalam perhitungan, akan dilakukan sedikit modifikasi aljabar dari bentuk taksiran BLUP yang telah diperoleh pada persamaan (3.12), menjadi bentuk lain, yaitu:

$$\left(T(\hat{\theta}_i)\right)(\sigma_v^2) = \mathbf{z_i^T}\hat{\boldsymbol{\beta}}(\sigma_v^2) + \hat{v}_i(\sigma_v^2)$$

$$= \mathbf{z_i^T}\hat{\boldsymbol{\beta}}(\sigma_v^2) + \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\left(\hat{\theta}_i - \mathbf{z_i^T}\hat{\boldsymbol{\beta}}(\sigma_v^2)\right)$$

$$= \mathbf{z_i^T}\hat{\boldsymbol{\beta}}(\sigma_v^2) + \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\hat{\theta}_i - \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\hat{\boldsymbol{\beta}}(\sigma_v^2)$$

$$= \mathbf{z_i^T}\hat{\boldsymbol{\beta}}(\sigma_v^2) + \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\hat{\theta}_i - \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\hat{\boldsymbol{\beta}}(\sigma_v^2) + \mathbf{z_i^T}\boldsymbol{\beta} - \mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\boldsymbol{\beta}$$

$$- \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\boldsymbol{\beta}$$

$$= \mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\hat{\theta}_i - \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\boldsymbol{\beta} + \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \mathbf{z_i^T}\boldsymbol{\beta} - \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$$

$$+ \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\boldsymbol{\beta}$$

$$= \mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) + \mathbf{z_i^T}\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right) - \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)$$

$$= \mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) + \left(\mathbf{z_i^T} - \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right) \tag{3.13}$$

Dengan $\theta_i = \mathbf{z_i^T}\boldsymbol{\beta} + v_i$, diperoleh MSE dari $\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2)$ sebagai berikut:

$$\mathrm{MSE}\left[\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2)\right]$$

$$= E\left[\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2) - \theta_i\right]^2$$

$$= E\left[\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2) - \left(\mathbf{z_i^T}\boldsymbol{\beta} + v_i\right)\right]^2$$

$$= E\left[\mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) + \left(\mathbf{z_i^T} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right) - \left(\mathbf{z_i^T}\widehat{\boldsymbol{\beta}} + v_i\right)\right]^2$$

$$= E\left[\left\{\mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) - \left(\mathbf{z_i^T}\boldsymbol{\beta} + v_i\right)\right\} + \left(\mathbf{z_i^T} - \frac{\sigma_v^2}{\left(\sigma_v^2 + \psi_i\right)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]^2$$

$$= E\left[\mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) - \left(\mathbf{z_i^T}\boldsymbol{\beta} + v_i\right)\right]^2 + E\left[\left(\mathbf{z_i^T} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]^2$$

$$+ 2cov\left[\underbrace{\left(\mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) - \left(\mathbf{z_i^T}\boldsymbol{\beta} + v_i\right)\right)}_{i}, \underbrace{\left(\mathbf{z_i^T} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)}_{ii}\right]$$

karena (i) dan (ii) independen (dibuktikan pada Lampiran 10), maka $cov[i, ii] = 0$.

$$= \underbrace{E\left[\mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) - \left(\mathbf{z_i^T}\boldsymbol{\beta} + v_i\right)\right]^2}_{g_1(\sigma_v^2)} + \underbrace{E\left[\left(\mathbf{z_i^T} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]^2}_{g_2(\sigma_v^2)}$$

Dengan demikian, bentuk MSE BLUP (Rao, 2003) yang diperoleh yaitu:

$$\mathrm{MSE}\left[\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2)\right] = g_1(\sigma_v^2) + g_2(\sigma_v^2) \tag{3.14}$$

dimana

$$g_1(\sigma_v^2) = \frac{\sigma_v^2\psi_i}{(\sigma_v^2 + \psi_i)}$$

$$g_2(\sigma_v^2) = \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \mathbf{z_i^T} \left(\frac{(\sigma_v^2 + \psi_i)}{\mathbf{z_i z_i^T}}\right) \mathbf{z_i}$$

(dibuktikan pada Lampiran 11).

3.6 Metode EBLUP pada Model Fay-Herriot

Berbeda dengan metode BLUP yang telah dijelaskan di subbab sebelumnya, pada penaksiran parameter dengan metode EBLUP, variansi pengaruh acak ($\sigma_{\boldsymbol{v}}^{\mathbf{2}}$) tidak diketahui nilainya, sehingga harus ditaksir dari data. Salah satu metode yang dapat digunakan adalah metode *Maximum Likelihood* (ML).

Dengan metode ML, diperlukan fungsi *likelihood* yang merupakan pdf dari $\hat{\theta}_i$. Namun, terlebih dahulu akan dicari mean dan variansi dari $\hat{\theta}_i$, yaitu:

$$\begin{aligned}
E(\hat{\theta}_i) &= E(\mathbf{z_i^T\beta} + v_i + e_i) \\
&= E(\mathbf{z_i^T\beta}) + E(v_i) + E(e_i) \\
&= \mathbf{z_i^T\beta} + 0 + 0 \text{ ; karena } E(v_i) = 0 \text{ dan } E(e_i) = 0 \\
&= \mathbf{z_i^T\beta}.
\end{aligned}$$

$$\begin{aligned}
var(\hat{\theta}_i) &= \mathrm{E}[(\hat{\theta}_i - \mathbf{z_i^T\beta})(\hat{\theta}_i - \mathbf{z_i^T\beta})^T] \\
&= E[(v_i + e_i)(v_i + e_i)^T] \\
&= E[(v_i + e_i)({v_i}^T + {e_i}^T)] \\
&= E[v_i{v_i}^T + v_i{e_i}^T + e_i{v_i}^T + e_i{e_i}^T] \\
&= E[v_i{v_i}^T] + E[v_i{e_i}^T] + E[e_i{v_i}^T] + E[e_i{e_i}^T] \\
&= \sigma_{\boldsymbol{v}}^{\mathbf{2}} + 0 + 0 + \psi_i \\
&= \sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i.
\end{aligned}$$

Karena $v_i$dan e_i berdistribusi normal, maka $\hat{\theta}_i$ juga berdistribusi normal, sehingga dapat ditulis sebagai $\hat{\theta}_i \sim N(\mathbf{z_i^T\beta}, (\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i))$. Sedangkan pdf dari $\hat{\theta}_i$ yaitu:

$$L(\hat{\boldsymbol{\beta}}, \sigma_{\boldsymbol{v}}^{\mathbf{2}}; \hat{\theta}_i) = f(\hat{\theta}_i)$$

$$= \frac{1}{(2\pi)^{1/2}(\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i)^{1/2}} \exp\left[-\frac{1}{2}\left((\hat{\theta}_i - \mathbf{z_i^T\beta})^T (\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i)^{-\mathbf{1}} (\hat{\theta}_i - \mathbf{z_i^T\beta})\right)\right] \tag{3.15}$$

Untuk mempermudah perhitungan, akan digunakan fungsi *log-likelihood*, yaitu:

$$\ln L\left(\widehat{\boldsymbol{\beta}}, \sigma_{\boldsymbol{v}}^{\mathbf{2}}; \hat{\theta}_i\right)$$
$$= -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\left[\ln(\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i) + \left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)^T (\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i)^{-\mathbf{1}}\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)\right] \quad (3.16)$$

Selanjutnya, fungsi *log-likelihood* tersebut diturunkan terhadap $\sigma_{\boldsymbol{v}}^{\mathbf{2}}$, dinotasikan dengan $\mathbf{s}(\widehat{\boldsymbol{\beta}}, \sigma_{\boldsymbol{v}}^{\mathbf{2}})$, sehingga untuk differensial terhadap elemen ke$-\,j$ diperoleh formula:

$$\mathbf{s}_j\left(\widehat{\boldsymbol{\beta}}, \sigma_v^2\right) = \frac{\partial \ln L\left(\widehat{\boldsymbol{\beta}}, \sigma_v^2; \hat{\theta}_i\right)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}}$$

$$= \frac{\partial\left(-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\left[\ln(\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i) + \left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)^T (\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i)^{-\mathbf{1}}\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)\right]\right)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}}$$

$$= \frac{\partial\left(-\frac{1}{2}\ln(2\pi)\right)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}} - \frac{1}{2}\frac{\partial\left(\ln(\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i)\right)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}}$$
$$- \frac{1}{2}\frac{\partial\left(\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)^T (\sigma_v^2 + \psi_i)^{-\mathbf{1}}\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)\right)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}}$$

$$= 0 - \frac{1}{2}\left[\frac{\partial\left(\ln(\sigma_v^2 + \psi_i)\right)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}}\right] - \frac{1}{2}\left[\frac{\partial\left(\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)^T (\sigma_v^2 + \psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)\right)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}}\right]$$

$$= -\frac{1}{2}\frac{1}{(\sigma_v^2 + \psi_i)} - \frac{1}{2}\left[\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)^T \left\{-\frac{1}{(\sigma_v^2 + \psi_i)^2}\underbrace{\frac{\partial(\sigma_v^2 + \psi_i)}{\partial \sigma_v^2}}_{=1}\right\}\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)\right]$$

$$= -\frac{1}{2}\frac{1}{(\sigma_v^2 + \psi_i)} + \frac{1}{2}\frac{\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)^T\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)}{(\sigma_v^2 + \psi_i)^2} \quad (3.17)$$

Berdasarkan prosedur metode *Maximum Likelihood*, akan dicari solusi dari persamaan berikut:

$$\frac{1}{2}\frac{1}{(\sigma_v^2 + \psi_i)} = \frac{1}{2}\frac{\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)^T\left(\hat{\theta}_i - \mathbf{z}_{\mathbf{i}}^{\mathbf{T}}\boldsymbol{\beta}\right)}{(\sigma_v^2 + \psi_i)^2}$$

Berdasarkan persamaan tersebut, terlihat bahwa $\sigma_{\boldsymbol{v}}^{\mathbf{2}}$ tidak dapat diselesaikan secara analitik. Oleh karena itu, taksiran $\sigma_{\boldsymbol{v}}^{\mathbf{2}}$ kemudian diselesaikan secara numerik, yaitu dengan *Scoring Algorithm*, dimana iterasi ke$-(\mathrm{a}+1)$:

$$\sigma_v^{2(a+1)} = \sigma_v^{2(a)} + \left[\mathcal{L}\left(\sigma_v^{2(a)}\right)\right]^{-1} \mathbf{s}(\widehat{\boldsymbol{\beta}}(\sigma_v^2)^{(a)}, \sigma_v^{2(a)}) \qquad (3.18)$$

(Rao, 2003) dimana

$$\mathcal{L}(\sigma_v^2) = \frac{\partial^2 \ln L(\widehat{\boldsymbol{\beta}}, \sigma_v^2; \hat{\theta}_i)}{\partial\sigma_v^2\,\partial\sigma_v^2}$$

$$= \frac{\partial}{\partial\sigma_v^2}\left(\frac{\partial \ln L(\widehat{\boldsymbol{\beta}}, \sigma_v^2; \hat{\theta}_i)}{\partial\sigma_v^2}\right)$$

$$= \frac{\partial}{\partial\sigma_v^2}\left(-\frac{1}{2}\frac{1}{(\sigma_v^2 + \psi_i)} + \frac{1}{2}\frac{(\hat{\theta}_i - \mathbf{z}_i^T\boldsymbol{\beta})^T(\hat{\theta}_i - \mathbf{z}_i^T\boldsymbol{\beta})}{(\sigma_v^2 + \psi_i)^2}\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial\sigma_v^2}\left(\frac{1}{(\sigma_v^2 + \psi_i)}\right) + \frac{1}{2}\frac{\partial}{\partial\sigma_v^2}\left(\frac{(\hat{\theta}_i - \mathbf{z}_i^T\boldsymbol{\beta})^2}{(\sigma_v^2 + \psi_i)^2}\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{(\sigma_v^2 + \psi_i)^2}\right) + \underbrace{\frac{1}{2}\frac{\partial}{\partial\sigma_v^2}\left[\frac{\hat{\theta}_i - \mathbf{z}_i^T\left(\frac{\mathbf{z}_i\hat{\theta}_i}{\mathbf{z}_i\mathbf{z}_i^T}\right)}{(\sigma_v^2 + \psi_i)}\right]^2}_{=0}$$

$$= \frac{1}{2}\frac{1}{(\sigma_v^2 + \psi_i)^2}.$$

dan

$$\mathbf{s}(\widehat{\boldsymbol{\beta}}, \sigma_v^2) = -\frac{1}{2}\frac{1}{\sigma_v^2 + \psi_i} + \frac{1}{2}\frac{(\hat{\theta}_i - \mathbf{z}_i^T\widehat{\boldsymbol{\beta}})^2}{(\sigma_v^2 + \psi_i)^2}$$

Proses iterasi tersebut berhenti jika $(\widehat{\sigma_v^2})^{(a+1)} \approx (\widehat{\sigma_v^2})^{(a)}$, dalam tugas akhir ini, ditetapkan $\left|(\widehat{\sigma_v^2})^{(a+1)} - (\widehat{\sigma_v^2})^{(a)}\right| < 10^{-5}$. Kemudian $(\widehat{\sigma_v^2})^{(a+1)}$dapat diambil sebagai taksiran dari σ_v^2. Selanjutnya $\widehat{\sigma_v^2}$ tersebut akan disubstitusikan ke dalam taksiran BLUP yang sebelumnya telah diperoleh pada persamaan (3.12).

Dengan demikian, taksiran EBLUP yang diperoleh untuk kombinasi linear $\mathbf{z}_i^T\boldsymbol{\beta} + v_i$, adalah:

$$\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) = \mathbf{z}_i^T\widehat{\boldsymbol{\beta}} + \frac{\widehat{\sigma_v^2}}{(\widehat{\sigma_v^2} + \psi_i)}(\hat{\theta}_i - \mathbf{z}_i^T\widehat{\boldsymbol{\beta}})$$
$$= \mathbf{z}_i^T\widehat{\boldsymbol{\beta}} + \hat{v}_i \qquad (3.19)$$

dengan

$$\widehat{\boldsymbol{\beta}}(\widehat{\sigma_v^2}) = (\mathbf{z}_i(\widehat{\sigma_v^2} + \psi_i)^{-1}\mathbf{z}_i^T)^{-1}\mathbf{z}_i(\widehat{\sigma_v^2} + \psi_i)^{-1}\hat{\theta}_i$$

$$\hat{v}_i(\widehat{\sigma_v^2}) = \frac{\widehat{\sigma_v^2}}{(\widehat{\sigma_v^2} + \psi_i)}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\hat{\boldsymbol{\beta}}(\widehat{\sigma_v^2})\right).$$

3.7 MSE EBLUP dan Taksiran MSE EBLUP pada Model Fay-Herriot

Untuk mengetahui seberapa baik taksiran EBLUP yang telah didapat pada subbab 3.6, akan dicari nilai MSE EBLUP nya, yaitu:

$$\begin{aligned}
&\mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)\right] \\
&= E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \theta_i\right]^2 \\
&= E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \theta_i + \left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2 \\
&= E\left[\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right) + \left(\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)\right]^2 \\
&= \underbrace{E\left[\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right)\right]^2}_{\mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]} + E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2 \\
&\quad +2cov\left[\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right), \left(\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)\right] \\
&= \mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right] + E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2 \\
&\quad +2cov\left[\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right), \left(\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)\right]
\end{aligned}$$

Karena $\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right)$ dan $\left(\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)$ independen (dibuktikan pada Lampiran 12), maka diperoleh:

$$\mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)\right] = \underbrace{\mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]}_{i} + \underbrace{E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2}_{ii} \qquad (3.20)$$

dimana $\mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]$ merupakan bentuk dari MSE BLUP.

Karena bagian (ii) pada persamaan (3.20) nilainya sangat sulit dicari, maka digunakan aproksimasi Taylor untuk menghitungnya. Kemudian didapat:

$$\begin{aligned}
E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^{\mathbf{2}} &\approx \frac{{\psi_i}^2}{(\sigma_v^2 + \psi_i)^3}\left[\boldsymbol{\mathcal{L}}(\boldsymbol{\sigma_v^2})\right]^{-1} \\
&= \frac{2{\psi_i}^2}{(\boldsymbol{\sigma_v^2} + \psi_i)} \\
&= g_3(\boldsymbol{\sigma_v^2}) \qquad (3.21)
\end{aligned}$$

dimana

$$\mathcal{L}_{\sigma_v^2}(\sigma_v^2) = var(\sigma_v^2) = 2(\sigma_v^2 + \psi_i)^2$$

(dibuktikan pada Lampiran 13).

Berdasarkan persamaan (3.14) dan (3.21), didapatkan aproksimasi kedua untuk MSE $\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2})\right]$ pada persamaan (3.20) sebagai berikut:

$$\text{MSE}\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2})\right] \approx g_1(\sigma_v^2) + g_2(\sigma_v^2) + g_3(\sigma_v^2) \quad (3.22)$$

dengan

$$g_1(\sigma_v^2) = \frac{\sigma_v^2\psi_i}{(\sigma_v^2 + \psi_i)}$$

$$g_2(\sigma_v^2) = \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \mathbf{z_i^T}\left(\frac{(\sigma_v^2 + \psi_i)}{\mathbf{z_i z_i^T}}\right)\mathbf{z_i}$$

$$g_3(\sigma_v^2) \approx \frac{2{\psi_i}^2}{(\sigma_v^2 + \psi_i)}.$$

Pada metode EBLUP, diasumsikan bahwa σ_v^2 tidak diketahui. Dengan demikian, MSE EBLUP yang diperoleh pada persamaan (3.22) juga masih bergantung pada nilai σ_v^2 yang tidak diketahui, sehingga nilai MSE EBLUP tidak dapat dicari. Oleh karena itu, perlu dilakukan penaksiran terhadap nilai MSE EBLUP tersebut.

Taksiran MSE EBLUP dapat diperoleh dengan beberapa cara. Cara pertama yaitu dengan mengganti σ_v^2 dengan $\widehat{\sigma_v^2}$ pada MSE BLUP di persamaan (3.14), sehingga diperoleh:

$$\text{mse}_N\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2})\right] = g_1(\widehat{\sigma_v^2}) + g_2(\widehat{\sigma_v^2}) \quad (3.23)$$

dan dikenal dengan *naive estimator*. Akan tetapi, *naive estimator* ini tidak mengikutsertakan penambahan variasi σ_v^2 yaitu bagian (ii) dari persamaan (3.22), sehingga tidak sesuai dengan MSE EBLUP. Kemudian digunakan cara lain untuk menaksir MSE EBLUP, yaitu dengan mensubstitusikan $\widehat{\sigma_v^2}$ ke σ_v^2 pada aproksimasi kedua MSE EBLUP di persamaan (3.22). Hasil taksiran MSE EBLUP yang didapat yaitu:

$$\text{mse}\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2})\right] = g_1(\widehat{\sigma_v^2}) + g_2(\widehat{\sigma_v^2}) + g_3(\widehat{\sigma_v^2}) \quad (3.24)$$

Namun, ternyata taksiran MSE EBLUP pada persamaan (3.24) tersebut bersifat bias terhadap MSE $\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)\right]$ (dibuktikan pada Lampiran 14). Oleh karena itu, perlu dicari taksiran dari aproksimasi MSE EBLUP yang *unbiased* untuk MSE $\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)\right]$ dan diperoleh:

$$
\begin{aligned}
E\left\{\text{mse}\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)\right]\right\} &= E\{g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + g_3\left(\widehat{\sigma_v^2}\right)\} \\
&= E[g_1\left(\widehat{\sigma_v^2}\right)] + E[g_2\left(\widehat{\sigma_v^2}\right)] + E[g_3\left(\widehat{\sigma_v^2}\right)] \\
&\approx \underbrace{g_1(\sigma_v^2) + c^T_{\widehat{\sigma_v^2}}(\sigma_v^2)\nabla g_1(\sigma_v^2) - g_3(\sigma_v^2)}_{E[g_1\left(\widehat{\sigma_v^2}\right)]} + \underbrace{g_2(\sigma_v^2)}_{E[g_2\left(\widehat{\sigma_v^2}\right)]} + \underbrace{g_3(\sigma_v^2)}_{E[g_3\left(\widehat{\sigma_v^2}\right)]}
\end{aligned}
$$

$$E\{g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + g_3\left(\widehat{\sigma_v^2}\right)\} \approx g_1(\sigma_v^2) + c^T_{\widehat{\sigma_v^2}}(\sigma_v^2)\nabla g_1(\sigma_v^2) - g_3(\sigma_v^2) + g_2(\sigma_v^2) + g_3(\sigma_v^2)$$

$$E\{g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + g_3\left(\widehat{\sigma_v^2}\right)\} - c^T_{\widehat{\sigma_v^2}}(\sigma_v^2)\nabla g_1(\sigma_v^2) + g_3(\sigma_v^2) \approx g_1(\sigma_v^2) + g_2(\sigma_v^2) + g_3(\sigma_v^2)$$

$$E\{g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + g_3\left(\widehat{\sigma_v^2}\right) - c^T_{\widehat{\sigma_v^2}}\left(\widehat{\sigma_v^2}\right)\nabla g_1\left(\widehat{\sigma_v^2}\right) + g_3\left(\widehat{\sigma_v^2}\right)\} \approx g_1(\sigma_v^2) + g_2(\sigma_v^2) + g_3(\sigma_v^2)$$

$$E\{g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + 2g_3\left(\widehat{\sigma_v^2}\right) - c^T_{\widehat{\sigma_v^2}}\left(\widehat{\sigma_v^2}\right)\nabla g_1\left(\widehat{\sigma_v^2}\right)\} \approx g_1(\sigma_v^2) + g_2(\sigma_v^2) + g_3(\sigma_v^2)$$

$$E\{g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + 2g_3\left(\widehat{\sigma_v^2}\right) - c^T_{\widehat{\sigma_v^2}}\left(\widehat{\sigma_v^2}\right)\nabla g_1\left(\widehat{\sigma_v^2}\right)\} \approx \text{MSE}[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)]$$

Dengan demikian, taksiran yang *unbiased* untuk MSE$\left[\left(T\left(\hat{\theta}_i\right)\right)\left(\widehat{\sigma_v^2}\right)\right]$ adalah:

$$\text{mse}\left[\left(T\left(\hat{\theta}_i\right)\right)\left(\widehat{\sigma_v^2}\right)\right] = g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + 2g_3\left(\widehat{\sigma_v^2}\right) - c_{\widehat{\sigma_v^2}}^T\left(\widehat{\sigma_v^2}\right)\nabla g_1\left(\widehat{\sigma_v^2}\right) \tag{3.25}$$

dengan

$$g_1\left(\widehat{\sigma_v^2}\right) = \frac{\widehat{\sigma_v^2}\psi_i}{\left(\widehat{\sigma_v^2} + \psi_i\right)}$$

$$g_2\left(\widehat{\sigma_v^2}\right) = \left(1 - \frac{\widehat{\sigma_v^2}}{\left(\widehat{\sigma_v^2} + \psi_i\right)}\right)^2 \mathbf{z_i^T}\left(\frac{\left(\widehat{\sigma_v^2} + \psi_i\right)}{\mathbf{z_i z_i^T}}\right)\mathbf{z_i}$$

$$g_3\left(\widehat{\sigma_v^2}\right) \approx \frac{2{\psi_i}^2}{\left(\widehat{\sigma_v^2} + \psi_i\right)}$$

$$c_{\widehat{\sigma_v^2}}^T\left(\widehat{\sigma_v^2}\right) = E\left[\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T\right]$$

(Rao, 2003).

BAB 4
CONTOH PENERAPAN

Pada bab 4 ini, akan dicari taksiran pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember. Penaksiran yang akan dilakukan yaitu penaksiran langsung dan penaksiran tidak langsung. Metode yang akan digunakan pada penaksiran tidak langsung adalah metode EBLUP. Selanjutnya, hasil dari kedua penaksiran tersebut akan dibandingkan.

4.1 Data

Data yang digunakan dalam tugas akhir ini adalah data Susenas 2008 dan data Potensi Desa (PODES) 2008 yang diperoleh dari data Badan Pusat Statistik (BPS). Secara umum, data terdiri dari data diskrit dan kontinu. Namun, dalam tugas akhir ini, akan digunakan data kontinu yaitu pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember. Pengeluaran rumah tangga per kapita merupakan salah satu alat untuk mengukur kesejahteraan masyarakat. Survei yang dilakukan oleh BPS, seperti Susenas 2008, dirancang untuk inferensial bagi daerah yang luas. Sedangkan untuk memperoleh informasi pada area yang lebih kecil (*small area*), seperti pada level desa/kelurahan, digunakan salah satu metode yang tepat yaitu *small area estimation* (SAE).

Kabupaten Jember merupakan salah satu kabupaten yang terletak di Propinsi Jawa Timur. Berdasarkan Laporan BPS 2007, Jawa Timur merupakan propinsi paling miskin di Pulau Jawa dan Kabupaten Jember merupakan satu di antara kota/kabupaten yang memiliki tingkat kemiskinan paling tinggi. Oleh karena itu, Kabupaten Jember dipilih sebagai daerah yang akan dibahas dalam tugas akhir ini.

Kabupaten Jember terdiri dari 247 desa, dimana 14,17% nya yaitu 35 desa terpilih sebagai sampel pada Susenas 2008. Setiap desa yang terpilih, diambil 14 sampai dengan 16 rumah tangga sebagai sampel, sehingga jumlah keseluruhan rumah tangga yang menjadi sampel adalah 549 rumah tangga. Jumlah rumah tangga yang menjadi sampel pada setiap desa tersebut sangat kecil jika dibandingkan dengan jumlah rumah tangga di masing-masing desa, yaitu berkisar antara 0,1% sampai dengan 1,67% (Matualage, 2012).

Dalam contoh aplikasi tugas akhir ini, akan dicari taksiran pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember dengan metode *Empirical Best Linear Unbiased Prediction* (EBLUP). Hasil taksiran pengeluaran rumah tangga per kapita dengan metode EBLUP yang telah diperoleh, kemudian dicari *Mean Squared Error* (MSE) nya dan dibandingkan dengan MSE dari hasil penaksiran langsung, untuk mengetahui metode penaksiran mana yang lebih baik.

Data yang digunakan sebagai penaksir langsung adalah rata-rata pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember. Penaksir langsung dan variansi sampel dihitung dari data pengeluaran rumah tangga per kapita yang terpilih menjadi sampel pada Susenas 2008, yaitu berjumlah 35 desa. Kemudian dengan variansi sampel tersebut dapat dihitung nilai MSE dari taksiran langsung.

Data variabel-variabel pendukung (*auxiliary variables*) diperoleh dari data PODES 2008. Berdasarkan ketersediaan data PODES, dipilih variabel-variabel pendukung yang berkaitan dengan faktor-faktor yang mempengaruhi pengeluaran rumah tangga (Sunandi, 2011). Variabel-variabel pendukung tersebut antara lain:

a. persentase keluarga yang menggantungkan hidupnya pada pertanian $(\mathbf{z}_1^{\mathrm{T}})$,
b. jumlah keluarga yang menerima askeskin dalam satu tahun terakhir$(\mathbf{z}_2^{\mathrm{T}})$,
c. jumlah keluarga pengguna listrik PLN $(\mathbf{z}_3^{\mathrm{T}})$,
d. jumlah keluarga yang mengenyam pendidikan SD, SMP, SMA, dan PT $(\mathbf{z}_4^{\mathrm{T}})$,
e. jumlah keluarga yang tinggal di pemukiman kumuh $(\mathbf{z}_5^{\mathrm{T}})$,
f. jumlah keluarga yang memiliki Surat Keterangan Tidak Mampu (SKTM) dan satu tahun terakhir $(\mathbf{z}_6^{\mathrm{T}})$,
g. jumlah keluarga yang pernah belajar di lembaga pendidikan dan ketrampilan $(\mathbf{z}_7^{\mathrm{T}})$,
h. jumlah keluarga yang memiliki anggota keluarga sebagai Tenaga Kerja Indonesia (TKI) $(\mathbf{z}_8^{\mathrm{T}})$.

Data variabel-variabel pendukung tersebut memiliki satuan yang berbeda-beda, sehingga perlu distandarisasi. Variabel-variabel yang telah distandarisasi diberikan pada Tabel 4.3.

4.2 Analisis Data

4.2.1 Penaksiran Langsung (*direct estimation*)

Penaksiran langsung pada pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember dilakukan dengan menggunakan *software Excel*.

Taksiran langsung pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember ($\hat{\theta}_i$) didapat dengan menghitung rata-rata pengeluaran rumah tangga per kapita di setiap desa, yaitu:

$$\hat{\theta}_i = \frac{\text{Jumlah pengeluaran rumah tangga per kapita di desa-i}}{\text{Jumlah rumah tangga di desa-i}}$$

Sedangkan MSE dan *standar error* (SE) yang merupakan akar kuadrat positif dari MSE taksiran langsung, dihitung dengan:

$$MSE(\hat{\theta}_i) = \frac{s_i^2}{n_i}, \text{ dengan } s_i^2 = \frac{1}{n_i - 1}\sum_i^{n_i}\left(\hat{\theta}_i - \bar{\hat{\theta}}_i\right)^2.$$

$$SE_{\hat{\theta}_i} = \sqrt{MSE(\hat{\theta}_i)}.$$

Hasil taksiran langsung pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember ($\hat{\theta}_i$) seperti terlihat pada Tabel 4.1 berikut.

Tabel 4. 1 Tabel Hasil Taksiran Langsung Pengeluaran Rumah Tangga per Kapita per Bulan di Desa di Kabupaten Jember (dalam Rupiah)

No. Desa	Taksiran Langsung ($\hat{\theta}_i$)	No. Desa	Taksiran Langsung ($\hat{\theta}_i$)
1	669.589	19	634.862
2	655.184	20	588.754
3	578.315	21	610.272
4	581.532	22	577.459
5	596.978	23	608.950
6	689.277	24	566.675
7	681.211	25	605.607
8	656.333	26	661.572
9	630.343	27	731.584
10	622.306	28	586.261
11	627.596	29	598.145
12	609.002	30	585.499
13	671.319	31	569.281
14	624.151	32	708.618
15	658.970	33	519.447
16	615.928	34	539.347
17	633.036	35	487.680
18	798.599		

Selanjutnya, akan diperiksa asumsi bahwa taksiran langsung pengeluaran rumah tangga per kapita yang telah diperoleh tersebut berdistribusi normal. Pengujian dilakukan dengan menggunakan uji Kolmogorov-Smirnov dan diberikan pada Lampiran 15.

Sedangkan SE dari taksiran langsung, diberikan pada Tabel 4.2 berikut.

Tabel 4. 2 Nilai SE Taksiran Langsung Pengeluaran Rumah Tangga per Kapita per Bulan di Desa di Kabupaten Jember (dalam Rupiah)

No. Desa	SE Taksiran Langsung	No. Desa	SE Taksiran Langsung
1	45.675	19	27.326
2	37.841	20	13.356
3	19.957	21	19.412
4	15.709	22	10.888
5	12.229	23	18.304
6	44.527	24	7.952
7	35.432	25	24.633
8	27.204	26	30.511
9	32.088	27	28.588
10	24.746	28	31.889
11	35.957	29	34.412
12	17.015	30	37.726
13	32.245	31	19.398
14	21.157	32	25.535
15	29.492	33	49.634
16	21.478	34	148.570
17	21.519	35	67.634
18	90.039		

Tabel 4. 1 menunjukkan bahwa terdapat tiga desa dengan pengeluaran rumah tangga per kapita yang cukup tinggi dibandingkan dengan desa-desa yang lain, yaitu desa ke-18, desa ke-27, dan desa ke-32, dimana desa yang memiliki pengeluaran rumah tangga per kapita tertinggi adalah desa ke-18, yaitu sebesar Rp 798.599,-. Hasil taksiran yang diperoleh dengan penaksiran langsung di atas, cenderung kurang akurat dikarenakan jumlah sampel yang dipilih sangat kecil untuk setiap desa yang terpilih sebagai *small area* yaitu 14 s.d 16 rumah tangga per desa. Hal ini mengakibatkan hasil taksiran pengeluaran rumah tangga per

kapita yang diperoleh dengan penaksiran langsung ini akan menghasilkan nilai *standar error* yang cukup besar (dapat dilihat pada Tabel 4. 2) karena sampel yang diambil tidak dapat menggambarkan pengeluaran per kapita yang sebenarnya di beberapa desa pada Kabupaten Jember. Oleh karena itu, akan dilakukan penaksiran tidak langsung (dengan metode EBLUP) yang menggunakan data variabel – variabel pendukung (*auxiliary variables*) dari data PODES 2008, dengan harapan bahwa taksiran pengeluaran per kapita yang dihasilkan menjadi lebih baik, yaitu menghasilkan nilai *standar error* yang lebih kecil.

4.2.2 Penaksiran Tidak Langsung (*indirect estimation*)

Penaksiran tidak langsung pada pengeluaran rumah tangga per kapita di desa di Kabupaten Jember dilakukan menggunakan *software R*, yaitu dengan program EBLUP ML.r, seperti yang diberikan pada Lampiran 16. Data $\mathbf{z}_i^T$ dimana i=1, 2, …, 8 yang diberikan pada Tabel 4.3 menyatakan variabel-variabel pendukung *small area* yang telah distandarisasi. Masing-masing variabel pendukung yang akan digunakan, sebelumnya telah dijelaskan pada subbab 4.1.

Tabel 4. 3 Data Variabel-Variabel Pendukung yang Berkaitan dengan Pengeluaran Rumah Tangga per Kapita per Bulan di Desa di Kabupaten Jember

z_1^T	z_2^T	z_3^T	z_4^T	z_5^T	z_6^T	z_7^T	z_8^T
-2.451	-5.899	0.629	2.268	1.347	-2.323	0.974	1.002
1.195	-3.821	-0.066	1.22	-3.512	-5.507	-1.77	-6.316
1.789	-0.987	1.841	-0.54	-0.979	-1.242	1.977	-2.361
1.252	4.113	-0.723	-5.311	-3.394	0.127	-0.629	-1.731
-1.076	-0.934	2.06	-0.852	-1.436	-1.735	2.171	0.621
-2.152	-0.731	0.013	2.882	1.31	-0.58	-1.882	-0.689
-2.071	1.265	1.676	1.058	0.995	-0.721	0.608	0.326
-2.478	0.737	-0.148	1.142	2.383	-1.917	-3.213	-4.941
-2.275	2.691	0.42	-0.916	0.226	-0.262	-1.111	4.172

5.19	-3.196	1.251	-1.198	1.592	-2.878	2.954	3.986
1.297	-2.447	-1.175	0.292	2.71	3.254	-0.331	-1.176
-5.257	0.353	-1.888	-0.664	-4.332	-1.755	-5.742	2.079
0.267	0.363	-0.58	-2.404	0.576	0.173	1.023	1.679
-3.393	0.081	0.014	1.072	1.904	-0.045	0.875	-2.309
-1.074	2.514	-1.361	0.778	-1.124	-1.558	-2.076	-1.717
-1.526	-3.074	0.356	1.766	4.879	5.635	-1.429	-0.887
3.628	-0.444	-4.077	-2.07	-0.981	0.068	-1.829	0.814
0.298	0.462	-1.793	1.204	1.34	-1.478	-0.467	-0.566
-0.054	0.419	0.401	1.107	3.097	-1.199	1.459	1.461
-1.257	1.306	-0.775	-2.279	-0.688	2.561	-2.842	-1.72
-0.649	-0.889	3.678	1.859	-2.461	-1.117	7.52	1.176
-1.079	-0.905	-0.154	1.198	0.608	-0.585	-0.698	-0.319
2.735	-2.33	-0.512	7.334	3.172	-4.758	1.464	-1.062
0.382	-0.631	0.666	1.622	0.175	1.308	-0.64	-0.702
-1.93	0.083	3.455	1.057	0.726	-1.991	-3.647	-1.507
-0.041	3.733	2.781	-2.382	0.503	-3.693	-0.265	-1.912
-1.091	1.608	1.84	-0.985	1.902	-3.752	-3.695	-1.744
-1.402	-1.055	0.291	-0.59	-0.858	-2.321	-3.884	-5.242
-0.824	-1.042	0.905	0.263	-0.262	1.631	-1.482	-0.199
-2.947	-1.951	1.807	-3.788	-0.696	-0.082	-1.108	-5.186
-3.119	1.182	1.016	-1.948	-4.661	3.38	-2.947	-2.926
1.228	-2.316	1.79	0.566	-2.825	4.108	-3.471	-4.077
-2.653	2.942	-0.742	0.665	-5.8	2.463	2.892	-1.735
-0.138	-5.381	1.35	0.893	-3.584	-0.792	8.38	-0.222
0.168	0.425	2.428	1.679	2.094	-0.976	0.797	0.391

Berdasarkan *output* yang dihasilkan oleh program EBLUP ML.r, diperoleh nilai taksiran pengaruh tetap ($\boldsymbol{\beta}$), yaitu $\widehat{\boldsymbol{\beta}} = \left(\hat{\beta}_1, \dots, \hat{\beta}_8\right)^T$ dan pengaruh acak (v_i), yaitu $\widehat{v_i}$, yang masing-masing diberikan pada Tabel 4.4 dan 4.5 di bawah ini.

Tabel 4. 4 Nilai Taksiran Pengaruh Tetap ($\hat{\boldsymbol{\beta}}$) dengan Metode EBLUP

Taksiran	EBLUP ML
$\hat{\beta}_1$	-63.875
$\hat{\beta}_2$	-47.021
$\hat{\beta}_3$	59.037
$\hat{\beta}_4$	-6.977
$\hat{\beta}_5$	3.049
$\hat{\beta}_6$	-51.072
$\hat{\beta}_7$	-17.284
$\hat{\beta}_8$	-59.079

Tabel 4. 5 Nilai Taksiran Pengaruh Acak ($\hat{v}_i$) dengan Metode EBLUP

No. Desa	Taksiran Pengaruh Acak ($\hat{v}_i$)	No. Desa	Taksiran Pengaruh Acak ($\hat{v}_i$)
1	166.389	19	674.222
2	-109.469	20	581.511
3	367.437	21	472.122
4	763.553	22	420.615
5	346.565	23	464.819
6	426.916	24	547.013
7	504.496	25	33.597
8	96.544	26	344.855
9	790.676	27	256.939
10	854.161	28	-66.699
11	746.070	29	489.578
12	343.443	30	-154.404
13	843.733	31	314.672
14	288.033	32	492.838
15	579.096	33	622.253
16	560.107	34	283.873
17	1.091.283	35	361.510
18	809.113		

Selanjutnya, akan diperiksa asumsi bahwa taksiran pengaruh acak *small area* yang telah diperoleh tersebut berdistribusi normal. Pengujian dilakukan dengan menggunakan uji Kolmogorov-Smirnov dan diberikan pada Lampiran 17.

Kemudian, akan dicari taksiran variansi pengaruh acak ($\hat{\sigma}_v^2$) secara numerik dengan menggunakan *scoring algorithm*. Dengan mengambil nilai awal $\hat{\sigma}_v^{2(0)}$=5.000.000, diperoleh nilai taksiran variansi pengaruh acak yaitu $\hat{\sigma}_v^2 = 280.108.000.000$.

Dengan menjalankan program EBLUP ML.r, dihasilkan nilai taksiran pengeluaran rumah tangga per kapita dengan metode EBLUP seperti yang diberikan pada Tabel 4.6 berikut.

Tabel 4. 6 Nilai Taksiran EBLUP dengan Program EBLUP ML.r

No. Desa	Taksiran EBLUP	No. Desa	Taksiran EBLUP
1	668.350	19	633.065
2	655.744	20	588.384
3	577.793	21	609.637
4	580.859	22	577.281
5	596.793	23	608.394
6	686.255	24	566.552
7	678.950	25	605.534
8	656.078	26	660.426
9	627.437	27	730.834
10	620.439	28	586.503
11	624.152	29	596.075
12	608.647	30	586.284
13	668.187	31	568.858
14	623.691	32	707.471
15	657.172	33	513.974
16	615.006	34	516.977
17	631.232	35	481.776
18	775.181		

Sedangkan nilai SE dari taksiran pengeluaran per kapita dengan metode EBLUP yang diperoleh, diberikan pada Tabel 4.7 di bawah ini.

Tabel 4. 7 Nilai SE Taksiran EBLUP dengan Program EBLUP ML.r

No. Desa	SE Taksiran EBLUP	No. Desa	SE Taksiran EBLUP
1	45.583	19	27.296
2	37.799	20	13.352
3	19.946	21	19.405
4	15.704	22	10.886
5	12.226	23	18.299
6	44.404	24	7.951
7	35.369	25	24.616
8	27.181	26	30.480
9	32.046	27	28.560
10	24.732	28	31.847
11	35.898	29	34.352
12	17.012	30	37.679
13	32.197	31	19.389
14	21.146	32	25.523
15	29.457	33	49.529
16	21.469	34	145.912
17	21.508	35	67.212
18	89.013		

Berdasarkan Tabel 4.6, desa-desa yang memiliki pengeluaran rumah tangga per kapita cukup tinggi adalah desa ke-18, desa ke-27, dan desa ke-32, dimana desa yang memiliki pengeluaran rumah tangga per kapita tertinggi adalah desa ke-18, yaitu sebesar Rp 775.181,-. Hal yang perlu diperhatikan adalah nilai *standar error* yang dihasilkan oleh penaksiran tidak langsung ini (dapat dilihat pada Tabel 4. 7) lebih kecil dibandingkan nilai *standar error* yang dihasilkan penaksiran langsung.

4.2.3 Perbandingan Penaksiran Langsung dan Penaksiran Tidak Langsung

Penaksiran langsung dan penaksiran tidak langsung yang telah dilakukan, masing-masing menghasilkan taksiran langsung dan MSE taksiran langsung, juga taksiran EBLUP dan taksiran MSE EBLUP. Berikut akan diberikan perbandingan antara taksiran langsung dan taksiran taksiran pengeluaran per kapita dengan metode EBLUP pada Tabel 4.8.

Tabel 4. 8 Tabel Perbandingan Hasil Penaksiran Langsung dan Penaksiran Tidak Langsung dengan Metode EBLUP

No. Desa	Taksiran Langsung	Taksiran EBLUP
1	669.589	668.350
2	655.184	655.744
3	578.315	577.793
4	581.532	580.859
5	596.978	596.793
6	689.277	686.255
7	681.211	678.950
8	656.333	656.078
9	630.343	627.437
10	622.306	620.439
11	627.596	624.152
12	609.002	608.647
13	671.319	668.187
14	624.151	623.691
15	658.970	657.172
16	615.928	615.006
17	633.036	631.232
18	798.599	775.181
19	634.862	633.065

20	588.754	588.384
21	610.272	609.637
22	577.459	577.281
23	608.950	608.394
24	566.675	566.552
25	605.607	605.534
26	661.572	660.426
27	731.584	730.834
28	586.261	586.503
29	598.145	596.075
30	585.499	586.284
31	569.281	568.858
32	708.618	707.471
33	519.447	513.974
34	539.347	516.977
35	487.680	481.776

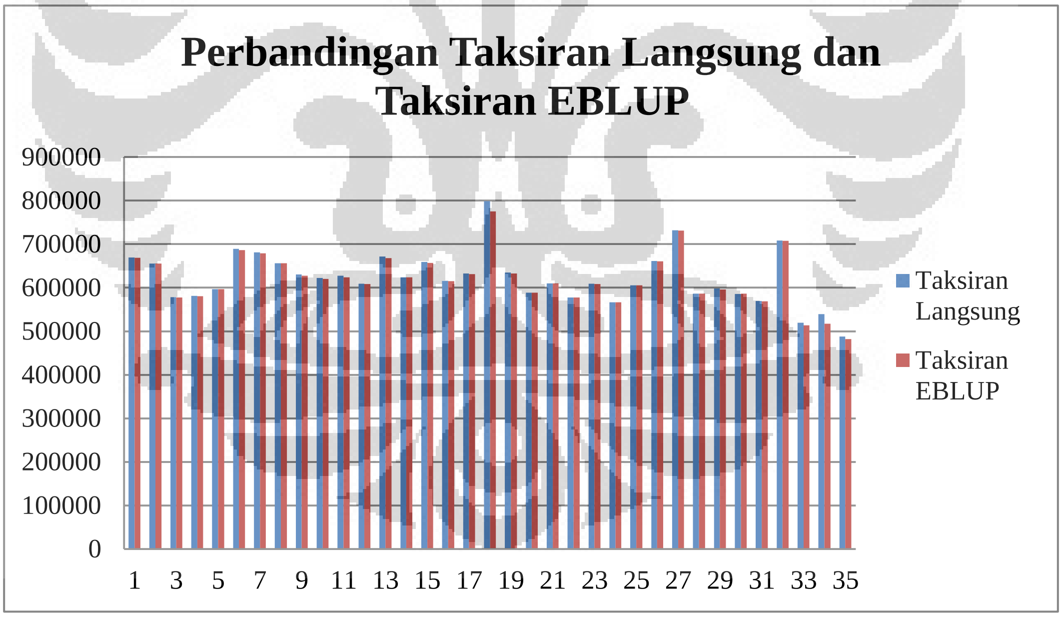
Sedangkan perbandingan antara SE dari taksiran langsung dan taksiran EBLUP, diberikan pada Tabel 4.9 berikut.

Tabel 4. 9 Tabel Perbandingan SE Hasil Penaksiran Langsung dan Penaksiran Tidak Langsung dengan Metode EBLUP

No. Desa	SE Taksiran Langsung	SE Taksiran EBLUP
1	45.675	45.230
2	37.841	37.637
3	19.957	19.904
4	15.709	15.688
5	12.229	12.216
6	44.527	43.924
7	35.432	35.122
8	27.204	27.090
9	32.088	31.886
10	24.746	24.678
11	35.957	35.666
12	17.015	17.001
13	32.245	32.011
14	21.157	21.102
15	29.492	29.321
16	21.478	21.435
17	21.519	21.466
18	90.039	85.201
19	27.326	27.182
20	13.356	13.339
21	19.412	19.378
22	10.888	10.879
23	18.304	18.281
24	7.952	7.948
25	24.633	24.552
26	30.511	30.362

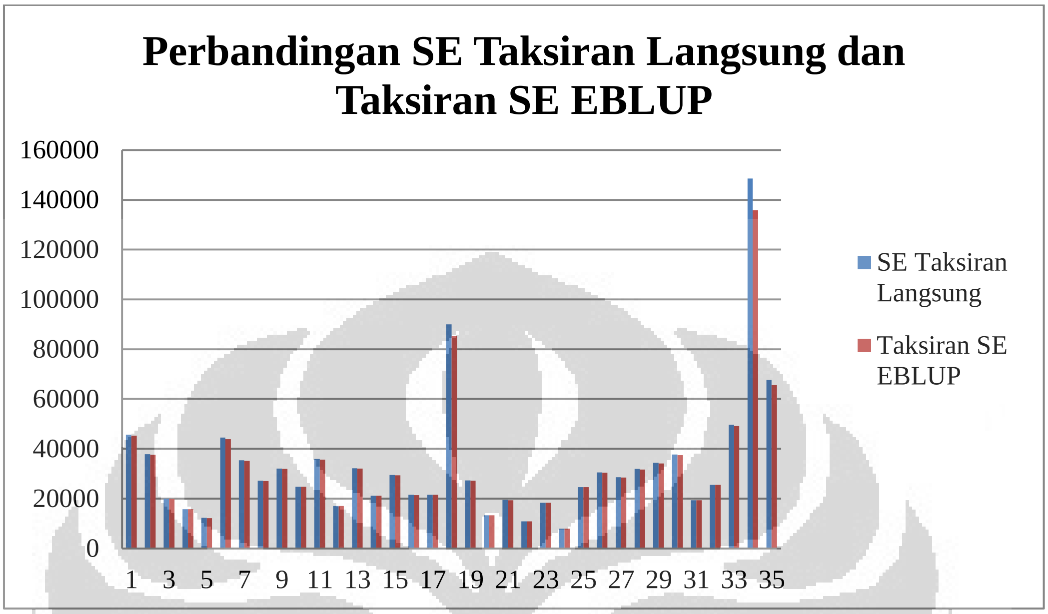
27	28.588	28.450
28	31.889	31.687
29	34.412	34.120
30	37.726	37.495
31	19.398	19.356
32	25.535	25.475
33	49.634	49.121
34	148.570	135.871
35	67.634	65.597

Grafik perbandingan taksiran langsung dan taksiran EBLUP diberikan pada Gambar 4.1 di bawah ini.



Gambar 4. 1 Grafik Perbandingan Taksiran Langsung dan Taksiran EBLUP

Sedangkan perbandingan antara SE taksiran langsung dan taksiran SE EBLUP disajikan dalam Gambar 4.2 berikut.



Gambar 4. 2 Grafik Perbandingan SE Taksiran Langsung dan Taksiran SE EBLUP

Berdasarkan Tabel 4. 8, didapatkan kedua penaksiran yaitu penaksiran langsung dan penaksiran tidak langsung sama-sam memperlihatkan bahwa terdapat tiga desa yang memiliki pengeluaran rumah tangga per kapita cukup tinggi. Desa-desa tersebut adalah desa ke-18, desa ke-27, dan desa ke-32, dengan pengeluaran rumah tangga per kapita tertinggi adalah desa ke-18, yaitu Rp 775.181,- (hasil penaksiran tidak langsung).

Sedangkan Tabel 4. 9 menunjukkan bahwa nilai *standar error* dari penaksiran tidak langsung dengan menggunakan metode EBLUP lebih kecil dibandingkan dengan *standar error* dari hasil penaksiran langsung. Oleh karena itu, penaksiran tidak langsung (dengan metode EBLUP) lebih baik daripada penaksiran langsung.

BAB 5
KESIMPULAN DAN SARAN

5.1 Kesimpulan

Small area merupakan gambaran suatu area geografis kecil atau subpopulasi kecil. Karena areanya yang kecil, penaksiran langsung pada *small area* akan menghasilkan *standar error* yang besar, sehingga diperlukan penaksiran tidak langsung. Penaksiran tidak langsung pada *small area* dapat dilakukan dengan beberapa metode, salah satunya adalah dengan metode *Empirical Best Linear Unbiased Prediction* (EBLUP). Metode EBLUP ini digunakan pada model Fay-Herriot untuk menaksir parameter *small area*. Taksiran parameter EBLUP yang dihasilkan, yaitu:

$$\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) = \mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}} + \hat{v}_i$$

dengan

$$\widehat{\boldsymbol{\beta}}(\widehat{\sigma_v^2}) \;= (\mathbf{z}_\mathbf{i}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T})^{-1}\mathbf{z}_\mathbf{i}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i$$

dan taksiran pengaruh acak, yaitu:

$$\hat{v}_i(\widehat{\sigma_v^2}) = \frac{\widehat{\sigma_v^2}}{(\widehat{\sigma_v^2} + \psi_i)}\left(\hat{\theta}_i - \mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\widehat{\sigma_v^2})\right).$$

Untuk mengatahui seberapa baik taksiran EBLUP yang diperoleh, perlu dicari nilai *Mean Squared Error* (MSE) EBLUP.

Nilai MSE EBLUP terdiri dari tiga komponen yaitu $g_1\left(\widehat{\sigma_v^2}\right)$, $g_2\left(\widehat{\sigma_v^2}\right)$, dan $g_3\left(\widehat{\sigma_v^2}\right)$, dimana komponen pertama dan kedua merupakan dua komponen yang sama seperti yang ada pada MSE BLUP. Sedangkan komponen ketiga merupakan faktor ketidakpastian dalam pemilihan taksiran variansi pengaruh acak *small area* $\left(\widehat{\sigma_v^2}\right)$. Nilai MSE EBLUP ini, masih mengandung σ_v^2 yang tidak diketahui, sehingga perlu dilakukan penaksiran pada MSE EBLUP. Taksiran MSE EBLUP yang *unbiased* terhadap MSE EBLUP diperoleh sebagai berikut:

$$\text{mse}\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)\right] = g_1\left(\widehat{\sigma_v^2}\right) + g_2\left(\widehat{\sigma_v^2}\right) + 2g_3\left(\widehat{\sigma_v^2}\right) - c_{\widehat{\sigma_v^2}}^T\left(\widehat{\sigma_v^2}\right)\nabla g_1\left(\widehat{\sigma_v^2}\right)$$

dengan

$$g_1\left(\widehat{\sigma_v^2}\right) = \frac{\widehat{\sigma_v^2}\psi_i}{\left(\widehat{\sigma_v^2} + \psi_i\right)}$$

$$g_2\left(\widehat{\sigma_v^2}\right) = \left(1 - \frac{\widehat{\sigma_v^2}}{\left(\widehat{\sigma_v^2} + \psi_i\right)}\right)^2 \mathbf{z_i^T}\left(\frac{\left(\widehat{\sigma_v^2} + \psi_i\right)}{\mathbf{z_i z_i^T}}\right)\mathbf{z_i}$$

$$g_3\left(\widehat{\sigma_v^2}\right) \approx \frac{2{\psi_i}^2}{\left(\widehat{\sigma_v^2} + \psi_i\right)}$$

$$c_{\widehat{\sigma_v^2}}^T\left(\widehat{\sigma_v^2}\right) = E\left[\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T\right].$$

Penaksiran tidak langsung dengan metode EBLUP menghasilkan nilai taksiran MSE yang lebih kecil dibandingkan MSE hasil penaksiran langsung. Oleh karena itu, penaksiran tidak langsung pengeluaran rumah tangga per kapita per bulan di desa di Kabupaten Jember dengan metode EBLUP pada model Fay-Herriot lebih baik dibandingkan dengan hasil penaksiran langsung.

5.2 Saran

Beberapa saran yang perlu diperhatikan, diantaranya:

1. Pada tugas akhir ini, model *small area* yang digunakan adalah salah satu kasus khusus dari model area level, yaitu model Fay-Herriot. Untuk penelitian selanjutnya dapat dilakukan pada model unit level.
2. Metode yang digunakan dalam tugas akhir ini adalah metode EBLUP. Untuk pembahasan berikutnya, dapat dilanjutkan dengan metode Spasial EBLUP (SEBLUP) yang telah memperhitungkan pengaruh spasial pada *small area*.
3. Pada tugas akhir ini, metode yang digunakan untuk menaksir variansi pengaruh acak *small area* adalah metode *Maximum Likelihood*. Metode lain yang dapat digunakan pada penelitian selanjutnya yaitu metode *moment* dan metode *Restricted Maximum Likelihood* (REML).
4. Tugas akhir ini bertujuan mencari taksiran MSE EBLUP pada model Fay-Herriot, dengan asumsi normal dari taksiran langsung, pengaruh acak *small area*, dan *sampling error* telah terpenuhi. Selanjutnya, dapat dipelajari mengenai taksiran MSE EBLUP yang *robust* terhadap asumsi normal.

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LAMPIRAN

Lampiran 1. Bukti Lemma 2.1

Lemma 2.1 Jika **A** matriks definit positif, maka **A** adalah matriks *nonsingular*

Bukti :

Misalkan **A** merupakan matriks definit positif, berukuran nxn.

Dengan menggunakan kontradiksi, dimisalkan **A** merupakan matriks yang *singular* atau $\text{rank}(\mathbf{A}) < n$, sehingga kolom dari **A** bergantung linear, maka terdapat vektor $\mathbf{x}^T$ sedemikian sehingga $\mathbf{Ax} = \mathbf{0}$, sehingga didapat:

$$\mathbf{x}^T\mathbf{Ax} = \mathbf{x}^T(\mathbf{Ax}) = \mathbf{x0} = 0.$$

Berdasarkan penjelasan tersebut, maka **A** bukan merupakan matriks definit positif. Dengan demikian, terbukti dengan menggunakan kontradiksi bahwa jika **A** merupakan matriks definit positif, maka **A** merupakan matriks *nonsingular*.

Lampiran 2. Bukti Teorema 2.1

Teorema 2.1

Misalkan matriks $\mathbf{M}$ berukuran $(p+q)x(p+q)$ yang dipartisi menjadi $\mathbf{M} = \left[\begin{array}{c|c}\mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D}\end{array}\right]$ dengan ukuran submatriks $\mathbf{A}_{\text{pxp}}$, $\mathbf{B}_{\text{pxq}}$, $\mathbf{C}_{\text{qxp}}$, $\mathbf{D}_{\text{qxq}}$ dan $\mathbf{A}$ *nonsingular* ($\mathbf{A}^{-1}$ada). $\mathbf{M}$ *nonsingular* jika dan hanya jika $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ *nonsingular*. Bentuk $\mathbf{M}^{-1}$ sebagai berikut:

$$\mathbf{M}^{-1} = \left[\begin{array}{c|c}\mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & \mathbf{D}'\end{array}\right]$$

$$= \begin{bmatrix}\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{E}^{-1}\end{bmatrix}$$

dengan $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$.

Bukti:

(i) Jika $\mathbf{M}$ *nonsingular* maka $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ *nonsingular*.

Misalkan $\mathbf{M}$ *nonsingular* atau $\mathbf{M}^{-1}$ ada, partisi $\mathbf{M}^{-1}$ menjadi $\left[\begin{array}{c|c}\mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & \mathbf{D}'\end{array}\right]$ dengan $\mathbf{A}'_{\text{pxp}}$, $\mathbf{B}'_{\text{pxq}}$, $\mathbf{C}'_{\text{qxp}}$, dan $\mathbf{D}'_{\text{qxq}}$, maka

$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}_{(p+q)x(p+q)}$ (1)

Matriks $\mathbf{I}$ dipartisi menjadi $\left[\begin{array}{c|c}\mathbf{I}' & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}''\end{array}\right]$ dengan $\mathbf{I}'$ adalah berukuran pxp dan $\mathbf{I}''$ adalah matriks qxq.

Berdasarkan persamaan (1) diperoleh:

$$\begin{bmatrix}\mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}'\end{bmatrix} = \begin{bmatrix}\mathbf{I}' & \mathbf{0} \\ \mathbf{0} & \mathbf{I}''\end{bmatrix} \quad (2)$$

Kemudian dari bentuk persamaan (2) akan diperoleh sistem persamaan berikut:

$$\mathbf{A}\mathbf{A}' + \mathbf{B}\mathbf{C}' = \mathbf{I}' \quad (3)$$

$$\mathbf{A}\mathbf{B}' + \mathbf{B}\mathbf{D}' = \mathbf{0} \quad (4)$$

$$\mathbf{C}\mathbf{A}' + \mathbf{D}\mathbf{C}' = \mathbf{0} \quad (5)$$

$$\mathbf{C}\mathbf{B}' + \mathbf{D}\mathbf{D}' = \mathbf{I}'' \quad (6)$$

Kalikan kedua ruas dengan $\mathbf{A}^{-1}$ pada persamaan (4), sehingga diperoleh:

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{B}' + \mathbf{A}^{-1}\mathbf{B}\mathbf{D}' = \mathbf{0}$$

Lampiran 2 (lanjutan)

$$\leftrightarrow \mathbf{IB}' + \mathbf{A}^{-1}\mathbf{BD}' = \mathbf{0}$$

$$\leftrightarrow \mathbf{B}' = -\mathbf{A}^{-1}\mathbf{BD}' \tag{7}$$

Substitusikan persamaan (7) ke persamaan (6), sehingga diperoleh:

$$\mathbf{CB}' + \mathbf{DD}' = \mathbf{I}''$$

$$\leftrightarrow \mathbf{C}(-\mathbf{A}^{-1}\mathbf{BD}') + \mathbf{DD}' = \mathbf{I}''$$

$$\leftrightarrow (-\mathbf{CA}^{-1}\mathbf{BD}') + \mathbf{DD}' = \mathbf{I}''$$

$$\leftrightarrow \underbrace{(\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B})}_{=\mathbf{E}}\mathbf{D}' = \mathbf{I}''$$

$$\leftrightarrow \mathbf{ED}' = \mathbf{I}''$$

$$\leftrightarrow \mathbf{D}' = \mathbf{E}^{-1} \tag{8}$$

dengan $\mathbf{E}$ merupakan matriks berukuran qxq yang *nonsingular*.

Jadi, terbukti bahwa jika $\mathbf{M}$ *nonsingular*, maka $\mathbf{E} = \mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}$ *nonsingular*.

(ii) Jika $\mathbf{E} = \mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}$ *nonsingular* maka $\mathbf{M}$ *nonsingular*.

Berdasarkan (i) diperoleh bahwa $\mathbf{D}' = \mathbf{E}^{-1}$ dan $\mathbf{B}' = -\mathbf{A}^{-1}\mathbf{BD}' = -\mathbf{A}^{-1}\mathbf{BE}^{-1}$. Selanjutnya, kalikan kedua ruas pada persamaan (3) dengan $\mathbf{A}^{-1}$ sehingga menjadi:

$$\mathbf{A}^{-1}\mathbf{AA}' + \mathbf{A}^{-1}\mathbf{BC}' = \mathbf{A}^{-1}$$

$$\leftrightarrow \mathbf{IA}' + \mathbf{A}^{-1}\mathbf{BC}' = \mathbf{A}^{-1}$$

$$\leftrightarrow \mathbf{A}' = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{BC}' \tag{9}$$

dengan mensubstitusikan persamaan (9) ke persamaan (5), diperoleh:

$$\mathbf{C}(\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{BC}') + \mathbf{DC}' = \mathbf{0}$$

$$\leftrightarrow (\mathbf{CA}^{-1} - \mathbf{CA}^{-1}\mathbf{BC}') + \mathbf{DC}' = \mathbf{0}$$

$$\leftrightarrow \mathbf{CA}^{-1} + \underbrace{(\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B})}_{=\mathbf{E}}\mathbf{C}' = \mathbf{0}$$

$$\leftrightarrow \mathbf{CA}^{-1} + \mathbf{EC}' = \mathbf{0}$$

$$\leftrightarrow \mathbf{EC}' = -\mathbf{CA}^{-1} \tag{10}$$

Kalikan kedua ruas pada persamaan (10) dengan $\mathbf{E}^{-1}$, sehingga diperoleh:

$$\mathbf{E}^{-1}\mathbf{EC}' = -\mathbf{E}^{-1}\mathbf{CA}^{-1}$$

$$\leftrightarrow \mathbf{IC}' = -\mathbf{E}^{-1}\mathbf{CA}^{-1}$$

Lampiran 2 (lanjutan)

$$\leftrightarrow \mathbf{C}' = -\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} \tag{11}$$

Selanjutnya, dengan mensubstitusikan persamaan (11) ke persamaan (9), diperoleh:

$$\mathbf{A}' = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(-\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1})$$

$$\leftrightarrow \mathbf{A}' = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} \tag{12}$$

Berdasarkan persamaan (12), (7), (11), dan (8) diperoleh:

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}' \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{E}^{-1} \end{bmatrix}$$

dengan $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$.

Jadi, terbukti bahwa jika $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ *nonsingular* maka $\mathbf{M}$ *nonsingular*.

(terbukti)

Lampiran 3. Bukti Teorema dan Lemma yang Berkaitan dengan Differensiasi Matriks

Bukti Teorema 2.2

Bentuk $\mathbf{a}^\mathrm{T}\mathbf{x}$ dapat dituliskan sebagai:

$$\mathbf{a}^\mathrm{T}\mathbf{x} = [\mathrm{a}_1 \quad \mathrm{a}_2 \quad \dots \quad \mathrm{a}_\mathrm{n}] \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \vdots \\ \mathrm{x}_\mathrm{n} \end{bmatrix} = [\mathrm{a}_1\mathrm{x}_1 + \mathrm{a}_2\mathrm{x}_2 + \cdots + \mathrm{a}_\mathrm{n}\mathrm{x}_\mathrm{n}]$$

sehingga

$$\frac{\partial(\mathbf{a}^\mathrm{T}\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial(\mathbf{a}^\mathrm{T}\mathbf{x})}{\partial \mathrm{x}_1} \\ \frac{\partial(\mathbf{a}^\mathrm{T}\mathbf{x})}{\partial \mathrm{x}_2} \\ \vdots \\ \frac{\partial(\mathbf{a}^\mathrm{T}\mathbf{x})}{\partial \mathrm{x}_\mathrm{n}} \end{bmatrix}^T = \begin{bmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \\ \vdots \\ \mathrm{a}_\mathrm{n} \end{bmatrix} = \mathbf{a}$$

(terbukti).

Bukti Teorema 2.2

Bentuk $\mathbf{x}^\mathrm{T}\mathbf{A}\mathbf{x}$ dapat dituliskan sebagai:

$$\mathbf{x}^\mathrm{T}\mathbf{A}\mathbf{x} = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \dots \quad \mathrm{x}_\mathrm{n}] \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \dots & \mathrm{a}_{1\mathrm{n}} \\ \vdots & \vdots & & \vdots \\ \mathrm{a}_{\mathrm{n}1} & \mathrm{a}_{\mathrm{n}2} & \dots & \mathrm{a}_{\mathrm{nn}} \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \vdots \\ \mathrm{x}_\mathrm{n} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{a}_{11}\mathrm{x}_1 + \mathrm{a}_{21}\mathrm{x}_2 + \cdots + \mathrm{a}_{\mathrm{n}1}\mathrm{x}_\mathrm{n} \\ \mathrm{a}_{12}\mathrm{x}_1 + \mathrm{a}_{22}\mathrm{x}_2 + \cdots + \mathrm{a}_{\mathrm{n}2}\mathrm{x}_\mathrm{n} \\ \vdots \\ \mathrm{a}_{1\mathrm{n}}\mathrm{x}_1 + \mathrm{a}_{2\mathrm{n}}\mathrm{x}_2 + \cdots + \mathrm{a}_{\mathrm{nn}}\mathrm{x}_\mathrm{n} \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \vdots \\ \mathrm{x}_\mathrm{n} \end{bmatrix}$$

$$= \begin{bmatrix} (\mathrm{a}_{11}\mathrm{x}_1 + \cdots + \mathrm{a}_{\mathrm{n}1}\mathrm{x}_\mathrm{n})\mathrm{x}_1 + (\mathrm{a}_{12}\mathrm{x}_1 + \cdots + \mathrm{a}_{\mathrm{n}2}\mathrm{x}_\mathrm{n})\mathrm{x}_2 + \cdots \\ +(\mathrm{a}_{1\mathrm{n}}\mathrm{x}_1 + \cdots + \mathrm{a}_{\mathrm{nn}}\mathrm{x}_\mathrm{n})\mathrm{x}_\mathrm{n} \end{bmatrix}$$

$$\frac{\partial(\mathbf{x}^\mathrm{T}\mathbf{A}\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial(\mathbf{x}^\mathrm{T}\mathbf{A}\mathbf{x})}{\partial \mathrm{x}_1} \\ \frac{\partial(\mathbf{x}^\mathrm{T}\mathbf{A}\mathbf{x})}{\partial \mathrm{x}_2} \\ \vdots \\ \frac{\partial(\mathbf{x}^\mathrm{T}\mathbf{A}\mathbf{x})}{\partial \mathrm{x}_\mathrm{n}} \end{bmatrix}$$

dengan

Lampiran 3 (lanjutan)

$$\frac{\partial(\mathbf{x^T A x})}{\partial \mathrm{x}_1} = 2\mathrm{a}_{11}\mathrm{x}_1 + \mathrm{a}_{21}\mathrm{x}_2 + \cdots + \mathrm{a}_{n1}\mathrm{x}_n + \mathrm{a}_{12}\mathrm{x}_2 + \cdots + \mathrm{a}_{1n}\mathrm{x}_n$$

$$\frac{\partial(\mathbf{x^T A x})}{\partial \mathrm{x}_2} = \mathrm{a}_{21}\mathrm{x}_1 + 2\mathrm{a}_{22}\mathrm{x}_2 + \cdots + \mathrm{a}_{n2}\mathrm{x}_n + \cdots + \mathrm{a}_{2n}\mathrm{x}_n$$

$$\vdots$$

$$\frac{\partial(\mathbf{x^T A x})}{\partial \mathrm{x}_n} = \mathrm{a}_{n1}\mathrm{x}_1 + \mathrm{a}_{n2}\mathrm{x}_2 + \mathrm{a}_{1n}\mathrm{x}_1 + \cdots + 2\mathrm{a}_{nn}\mathrm{x}_n$$

sehingga dapat dituliskan sebagai:

$$\frac{\partial(\mathbf{x^T A x})}{\partial \mathbf{x}} = \begin{bmatrix} 2\mathrm{a}_{11}\mathrm{x}_1 + \mathrm{a}_{21}\mathrm{x}_2 + \cdots + \mathrm{a}_{n1}\mathrm{x}_n + \mathrm{a}_{12}\mathrm{x}_2 + \cdots + \mathrm{a}_{1n}\mathrm{x}_n \\ \mathrm{a}_{21}\mathrm{x}_1 + 2\mathrm{a}_{22}\mathrm{x}_2 + \cdots + \mathrm{a}_{n2}\mathrm{x}_n + \cdots + \mathrm{a}_{2n}\mathrm{x}_n \\ \vdots \\ \mathrm{a}_{n1}\mathrm{x}_1 + \mathrm{a}_{n2}\mathrm{x}_2 + \mathrm{a}_{1n}\mathrm{x}_1 + \cdots + 2\mathrm{a}_{nn}\mathrm{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{a}_{11} + \mathrm{a}_{11} & \mathrm{a}_{21} + \mathrm{a}_{12} & \ldots & \mathrm{a}_{n1} + \mathrm{a}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{a}_{n1} + & \mathrm{a}_{n2} & \ldots & \mathrm{a}_{nn} + \mathrm{a}_{nn} \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \vdots \\ \mathrm{x}_n \end{bmatrix}$$

$$= \left(\begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{21} & \ldots & \mathrm{a}_{n1} \\ \mathrm{a}_{12} & \mathrm{a}_{22} & \ldots & \mathrm{a}_{n2} \\ \vdots & \vdots & & \vdots \\ \mathrm{a}_{1n} & \mathrm{a}_{2n} & \ldots & \mathrm{a}_{nn} \end{bmatrix} + \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \ldots & \mathrm{a}_{1n} \\ \mathrm{a}_{121} & \mathrm{a}_{22} & \ldots & \mathrm{a}_{n2n} \\ \vdots & \vdots & & \vdots \\ \mathrm{a}_{n1} & \mathrm{a}_{n2} & \ldots & \mathrm{a}_{nn} \end{bmatrix} \right) \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \vdots \\ \mathrm{x}_n \end{bmatrix}$$

$$= (\mathbf{A^T} + \mathbf{A})\mathbf{x}$$

$$= \mathbf{A^T x} + \mathbf{Ax}$$

Karena A matriks simetris, dimana $\mathbf{A^T} = \mathbf{A}$, maka didapat

$$\frac{\partial(\mathbf{x^T A x})}{\partial \mathbf{x}} = \mathbf{Ax} + \mathbf{Ax} = \mathbf{2Ax} \qquad \text{(terbukti).}$$

Bukti Lemma 2.2

Akan ditunjukkan bahwa

$$\frac{\partial(a\mathbf{A} + b\mathbf{B})}{\partial x_j} = a\frac{\partial \mathbf{A}}{\partial x_j} + b\frac{\partial \mathbf{B}}{\partial x_j}$$

dengan $a\frac{\partial \mathbf{A}}{\partial \mathrm{x_j}} = \begin{bmatrix} a\frac{\partial \mathrm{a}_{11}}{\partial \mathrm{x_j}} & a\frac{\partial \mathrm{a}_{12}}{\partial \mathrm{x_j}} & \ldots & a\frac{\partial \mathrm{a}_{1n}}{\partial \mathrm{x_j}} \\ \vdots & \vdots & & \vdots \\ a\frac{\partial \mathrm{a}_{n1}}{\partial \mathrm{x_j}} & a\frac{\partial \mathrm{a}_{n2}}{\partial \mathrm{x_j}} & \ldots & a\frac{\partial \mathrm{a}_{nn}}{\partial \mathrm{x_j}} \end{bmatrix}$ dan $b\frac{\partial \mathbf{B}}{\partial \mathrm{x_j}} = \begin{bmatrix} b\frac{\partial \mathrm{b}_{11}}{\partial \mathrm{x_j}} & b\frac{\partial \mathrm{b}_{12}}{\partial \mathrm{x_j}} & \ldots & b\frac{\partial \mathrm{b}_{1n}}{\partial \mathrm{x_j}} \\ \vdots & \vdots & & \vdots \\ b\frac{\partial \mathrm{b}_{n1}}{\partial \mathrm{x_j}} & b\frac{\partial \mathrm{b}_{n2}}{\partial \mathrm{x_j}} & \ldots & b\frac{\partial \mathrm{b}_{nn}}{\partial \mathrm{x_j}} \end{bmatrix}$

Bukti:

Lampiran 3 (lanjutan)

Berdasarkan sifat operator linear pada turunan, karena $\frac{\partial(a\mathbf{A})}{\partial x_j}$ dan $\frac{\partial(b\mathbf{B})}{\partial x_j}$ ada, maka $\frac{\partial(a\mathbf{A}+b\mathbf{B})}{\partial x_j}$ juga ada, dan $\frac{\partial(a\mathbf{A}+b\mathbf{B})}{\partial x_j} = \frac{\partial(a\mathbf{A})}{\partial x_j} + \frac{\partial(b\mathbf{B})}{\partial x_j} = a\frac{\partial \mathbf{A}}{\partial x_j} + b\frac{\partial \mathbf{B}}{\partial x_j}$ (terbukti).

Akan ditunjukkan bahwa

$$\frac{\partial(\mathbf{AB})}{\partial x_j} = \mathbf{A}\frac{\partial \mathbf{B}}{\partial x_j} + \frac{\partial \mathbf{A}}{\partial x_j}\mathbf{B}$$

Bukti:

Persamaan $\frac{\partial(\mathbf{AB})}{\partial x_j} = \mathbf{A}\frac{\partial \mathbf{B}}{\partial \boldsymbol{x_j}} + \frac{\partial \mathbf{A}}{\partial \boldsymbol{x_j}}\mathbf{B}$ yang akan dibuktikan dapat dituliskan sebagai $\frac{\partial}{\partial x_j}[\mathbf{AB}]_{ij} = \left[\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{B} + \mathbf{A}\frac{\partial \mathbf{B}}{\partial x_j}\right]_{ij}$.

Karena $[\mathbf{AB}]_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, maka:

$$\begin{aligned}\frac{\partial}{\partial x_j}[\mathbf{AB}]_{ij} &= \frac{\partial}{\partial x_j}\sum_{k=1}^{n} a_{ik}b_{kj}\\ &= \sum_{k=1}^{n}\frac{\partial}{\partial x_j}(a_{ik}b_{kj})\\ &= \sum_{k=1}^{n}\left(\frac{\partial a_{ik}}{\partial x_j}b_{kj} + \frac{\partial b_{kj}}{\partial x_j}a_{ik}\right)\\ &= \left[\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{B} + \mathbf{A}\frac{\partial \mathbf{B}}{\partial x_j}\right]_{ij} \qquad \text{(terbukti).}\end{aligned}$$

Bukti Lemma 2.3

Akan ditunjukkan bahwa

$$\frac{\partial \mathbf{A}^{-1}}{\partial x_j} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1}$$

Bukti:

Berdasarkan definisi invers matriks, diketahui bahwa

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

ketika kedua ruas didifferensialkan, menjadi

$$\frac{\partial \mathbf{A}^{-1}}{\partial x_j}\mathbf{A} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j} = \mathbf{0}$$

dengan memindah ruas, didapat

Lampiran 3 (lanjutan)

$$\frac{\partial \mathbf{A}^{-1}}{\partial x_j}\mathbf{A} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}$$

jika kedua ruas dikalikan dengan $\mathbf{A}^{-1}$, maka menjadi

$$\frac{\partial \mathbf{A}^{-1}}{\partial x_j} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1}$$ (terbukti).

Selanjutnya, akan ditunjukkan bahwa:

$$\frac{\partial^2 \mathbf{A}^{-1}}{\partial x_i \partial x_j} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i \partial x_j}\mathbf{A}^{-1} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}.$$

Bukti:

Dari pembuktian $\frac{\partial \mathbf{A}^{-1}}{\partial x_j}$ didapat juga $\frac{\partial \mathbf{A}}{\partial x_j} = -\mathbf{A}\frac{\partial \mathbf{A}^{-1}}{\partial x_j}\mathbf{A}$.

Berdasarkan definisi invers matriks, diketahui bahwa

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

ketika kedua ruas didifferensialkan, menjadi

$$\frac{\partial \mathbf{A}^{-1}}{\partial x_j}\mathbf{A} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j} = \mathbf{0}$$

Jika persamaan tersebut didifferensialkan terhadap x_i , menjadi

$$\frac{\partial}{\partial x_j}\left(\frac{\partial \mathbf{A}^{-1}}{\partial x_j}\right)\mathbf{A} + \frac{\partial \mathbf{A}^{-1}}{\partial x_j}\frac{\partial \mathbf{A}}{\partial x_i} + \frac{\partial \mathbf{A}^{-1}}{\partial x_i}\frac{\partial \mathbf{A}}{\partial x_j} + \mathbf{A}^{-1}\frac{\partial}{\partial x_i}\left(\frac{\partial \mathbf{A}}{\partial x_j}\right) = \mathbf{0}$$

Dengan menggunakan hasil yang diperoleh dari pembuktian $\frac{\partial \mathbf{A}^{-1}}{\partial x_j}$, maka

$$\frac{\partial}{\partial x_j}\left(\frac{\partial \mathbf{A}^{-1}}{\partial x_j}\right)\mathbf{A} = -\frac{\partial \mathbf{A}^{-1}}{\partial x_i}\frac{\partial \mathbf{A}}{\partial x_j} - \frac{\partial \mathbf{A}^{-1}}{\partial x_j}\frac{\partial \mathbf{A}}{\partial x_i} - \mathbf{A}^{-1}\frac{\partial}{\partial x_i}\left(\frac{\partial \mathbf{A}}{\partial x_j}\right)$$

$$= \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i} - \mathbf{A}^{-1}\frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_j}$$

$$\frac{\partial^2 \mathbf{A}^{-1}}{\partial x_i \partial x_j} = \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1} + \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1} - \mathbf{A}^{-1}\frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_j}\mathbf{A}^{-1}$$

(terbukti).

Lampiran 3 (lanjutan)

Bukti Lemma 2.4

Akan ditunjukkan bahwa

$$\frac{\partial \det(\mathbf{A})}{\partial x_j} = tr\left[adj(\mathbf{A})\frac{\partial \mathbf{A}}{\partial x_j}\right].$$

Bukti:

$\det(\mathbf{A})$ dapat juga dinyatakan sebagai fungsi dari elemen-elemen matriks **A**, yaitu:

$$\det(\mathbf{A}) = f(a_{11}, a_{12}, \dots, a_{nn})$$

Berdasarkan aturan rantai, $\frac{\partial \det(\mathbf{A})}{\partial x_j}$ dapat dituliskan sebagai

$$\frac{\partial \det(\mathbf{A})}{\partial x_j} = \frac{\partial f(a_{11}, a_{12}, \dots, a_{nn})}{\partial x_j}$$

$$= \frac{\partial f(a_{11}, a_{12}, \dots, a_{nn})}{\partial a_{11}}\frac{\partial a_{11}}{\partial x_j} + \frac{\partial f(a_{11}, a_{12}, \dots, a_{nn})}{\partial a_{12}}\frac{\partial a_{12}}{\partial x_j} + \cdots$$

$$+ \frac{\partial f(a_{11}, a_{12}, \dots, a_{nn})}{\partial a_{nn}}\frac{\partial a_{nn}}{\partial x_j}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial f(a_{11}, a_{12}, \dots, a_{nn})}{\partial a_{ik}}\frac{\partial a_{ik}}{\partial x_j}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial \det(\mathbf{A})}{\partial a_{ik}}\frac{\partial a_{ik}}{\partial x_j}$$

Akan ditunjukkan bahwa $\frac{\partial \det(\mathbf{A})}{\partial a_{ik}} = c_{ik}$.

Berdasarkan ekspansi baris ke$-i$, determinan dari matriks **A** dapat dinyatakan sebagai $\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik}c_{ik}$, dimana $c_{ik} = \left(adj(a_{ik})\right)^T$ adalah kofaktor dari a_{ik}. berdasarkan aturan rantai, maka diperoleh

$$\frac{\partial \det(\mathbf{A})}{\partial a_{ik}} = \frac{\partial}{\partial a_{ik}}\left(\sum_{l=1}^{n} a_{il}c_{il}\right)$$

$$= \sum_{l=1}^{n}\frac{\partial}{\partial a_{ik}}(a_{il}c_{il})$$

$$= \sum_{l=1}^{n}\frac{\partial a_{il}}{\partial a_{ik}}c_{il} + \sum_{l=1}^{n}a_{il}\frac{\partial c_{il}}{\partial a_{ik}}$$

Lampiran 3 (lanjutan)

Karena c_{il} bukan fungsi dalam a_{ik}, maka $\frac{\partial c_{il}}{\partial a_{ik}} = 0$, sehingga diperoleh

$$\frac{\partial \det(\mathbf{A})}{\partial a_{ik}} = \sum_{l=1}^{n} \frac{\partial a_{il}}{\partial a_{ik}} c_{il}$$

dimana $\frac{\partial a_{il}}{\partial a_{ik}} = \pi_{kl} \begin{cases} 0\,; k \neq l \\ 1\,; k = l \end{cases}$, maka

$$\begin{aligned} \frac{\partial \det(\mathbf{A})}{\partial a_{ik}} &= \sum_{l=1}^{n} c_{il}\, \pi_{kl} \\ &= c_{ik} \end{aligned}$$

sehingga diperoleh:

$$\begin{aligned} \frac{\partial \det(\mathbf{A})}{\partial x_j} &= \sum_{i=1}^{n} \sum_{k=1}^{n} \underbrace{\frac{\partial \det(\mathbf{A})}{\partial a_{ik}}}_{=c_{ik}} \underbrace{\frac{\partial a_{ik}}{\partial x_j}}_{=\frac{\partial \mathbf{A}}{\partial x_j}} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} c_{ik} \frac{\partial \mathbf{A}}{\partial x_j} \end{aligned}$$

Karena $c_{ik} = \left(adj(a_{ik})\right)^T$ dimana $i = 1, \dots, n$ dan $k = 1, \dots, n$, maka c_{ik} juga dapat dituliskan sebagai $c_{ik} = \left(adj(\mathbf{A})\right)^T$, sehingga diperoleh:

$$= tr\left[\left(adj(\mathbf{A})\right)^T \frac{\partial \mathbf{A}}{\partial x_j}\right]$$

Karena $tr\left[\left(adj(\mathbf{A})\right)^T\right] = tr[adj(\mathbf{A})]$, maka didapat:

$$\frac{\partial \det(\mathbf{A})}{\partial x_j} = tr\left[adj(\mathbf{A}) \frac{\partial \mathbf{A}}{\partial x_j}\right]$$

(terbukti).

Bukti Lemma 2.5

Akan ditunjukkan bahwa

$$\frac{\partial \log \det(\mathbf{A})}{\partial x_j} = tr\left[\mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial x_j}\right].$$

Bukti:

Berdasarkan Lemma 2.4, didapat bahwa $\frac{\partial \det(\mathbf{A})}{\partial a_{ik}} = tr\left[adj(\mathbf{A}) \frac{\partial \mathbf{A}}{\partial a_{ik}}\right]$, sehingga

Lampiran 3 (lanjutan)

$$\frac{\partial \log \det(\mathbf{A})}{\partial x_j} = \frac{1}{|\mathbf{A}|}\frac{\partial \det(\mathbf{A})}{\partial x_j}$$

$$= \frac{1}{|\mathbf{A}|} tr\left[adj(\mathbf{A})\frac{\partial \mathbf{A}}{\partial x_j}\right]$$

$$= tr\left[\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\right]. \qquad \text{(terbukti).}$$

Selanjutnya, akan ditunjukkan bahwa

$$\frac{\partial^2 \log \det(\mathbf{A})}{\partial x_i \partial x_j} = tr\left(\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i \partial x_j}\right) - tr\left(\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\right).$$

Bukti:

Karena $d(tr(\mathbf{A})) = d(tr(\mathbf{A}))$, maka

$$\frac{\partial^2 \log \det(\mathbf{A})}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(tr\left[\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\right]\right)$$

$$= tr\left(\frac{\partial}{\partial x_i}\left[\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\right]\right)$$

$$= tr\left(\frac{\partial \mathbf{A}^{-1}}{\partial x_i}\frac{\partial \mathbf{A}}{\partial x_j} + \mathbf{A}^{-1}\frac{\partial^2 \mathbf{A}}{\partial x_j \partial x_i}\right)$$

$$= tr\left(-\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j} + \mathbf{A}^{-1}\frac{\partial^2 \mathbf{A}}{\partial x_j \partial x_i}\right)$$

$$= tr\left(-\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_i}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\right) + tr\left(\mathbf{A}^{-1}\frac{\partial^2 \mathbf{A}}{\partial x_j \partial x_i}\right)$$

(terbukti).

Lampiran 4. Bukti $(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t})$ dan $\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)$ Independen

Akan ditunjukkan bahwa $(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t})$ dan $\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)$ independen. $(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t})$ dan $\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)$ dikatakan independen jika $cov\big[(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t}),\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)\big]=0.$

$$cov\big[(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t}),\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)\big]=$$

$$E\left[(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t})-\underbrace{E(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y})-\boldsymbol{t})}_{b}\right][\mathbf{d}^{\boldsymbol{T}}\underbrace{\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)}_{\boldsymbol{a}}-E(\mathbf{d}^{\boldsymbol{T}}\underbrace{\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}\big)}_{\boldsymbol{a}})]^T$$

Berikut adalah penjabarannya.

a. $\widehat{\boldsymbol{\beta}}(\delta)=(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{y}$

$$=(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)(\mathbf{X}\boldsymbol{\beta}+\mathbf{Zv}+\mathbf{e})$$
$$=(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X}\boldsymbol{\beta}+(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})$$
$$=\boldsymbol{\beta}+(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)(\mathbf{Zv}+\mathbf{e})$$
$$\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})-\boldsymbol{\beta}=(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)(\mathbf{Zv}+\mathbf{e})$$

b. $E[(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta},\boldsymbol{\beta},\mathrm{y})-\boldsymbol{t})]=E[\boldsymbol{\lambda}^T\boldsymbol{\beta}+\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)(\mathrm{y}-\mathrm{X}\boldsymbol{\beta})-(\boldsymbol{\lambda}^T\boldsymbol{\beta}+\boldsymbol{\omega}^T\mathrm{v})]$

$$=E[\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})(\mathbf{Zv}+\mathrm{e})-\boldsymbol{\omega}^T\mathrm{v}]$$
$$=E[\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{Zv}+\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{e}-\boldsymbol{\omega}^T\mathrm{v}]$$
$$=\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{Z}E[\mathrm{v}]+\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})E[\mathbf{e}]-\boldsymbol{\omega}^TE[\mathbf{v}]$$
$$=0\ ;\ \text{karena } E[\mathbf{v}]=0 \text{ dan } E[\mathbf{e}]=0.$$

c. $E\big[\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\delta)-\boldsymbol{\beta}\big)\big]=E[\boldsymbol{\lambda}^T-\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-1}(\delta)\mathrm{X}\big(\widehat{\boldsymbol{\beta}}(\delta)-\boldsymbol{\beta}\big)]$

$$=E[\boldsymbol{\lambda}^T-\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X}\Big((\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\Big)]$$
$$=E[\boldsymbol{\lambda}^T-\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X}\begin{pmatrix}(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{Zv}\\+(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{e}\end{pmatrix}]$$
$$=E\begin{bmatrix}(\boldsymbol{\lambda}^T-\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{Zv}\\+(\boldsymbol{\lambda}^T-\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{e}\end{bmatrix}$$
$$=(\boldsymbol{\lambda}^T-\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{Z}E[\mathbf{v}]$$
$$+(\boldsymbol{\lambda}^T-\boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})E[\mathbf{e}]$$

Lampiran 4 (lanjutan)

$= 0$; karena $E[\mathbf{v}] = 0$ dan $E[\mathbf{e}] = 0$.

Dari persamaan (a), (b), dan (c), maka diperoleh:

$$cov\left[(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t}), \mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\big)\right]$$

$$= E[(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t}) - E(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t})]$$
$$[\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\delta) - \boldsymbol{\beta}\big) - E(\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\delta) - \boldsymbol{\beta}\big))]^{T}$$
$$= E[(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t})][\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\big)]^{T}$$
$$= E[\boldsymbol{\lambda}^{T}\boldsymbol{\beta} + \boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) - (\boldsymbol{\lambda}^{T}\boldsymbol{\beta} + \boldsymbol{\omega}^{T}\mathbf{v})]$$
$$[(\boldsymbol{\lambda}^{T} - \boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})\big(\widehat{\boldsymbol{\beta}}(\delta) - \boldsymbol{\beta}\big)]^{T}$$
$$= E[\boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) - \boldsymbol{\omega}^{T}\mathbf{v}]$$
$$[(\boldsymbol{\lambda}^{T} - \boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-1}(\delta)\mathbf{X})(\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Z}\mathbf{v} + \mathbf{e})]^{T}$$
$$= E[\boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^{T}\mathbf{v}]$$
$$[(\mathbf{Z}\mathbf{v} + \mathbf{e})^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X}((\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})^{-1})^{\boldsymbol{T}}(\lambda^{T} - \boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\boldsymbol{\delta})\mathbf{X})^{T}]$$
$$= E[(\mathbf{a}^{\boldsymbol{T}}(\mathbf{Z}\mathbf{v} + \mathbf{e}) - \boldsymbol{\omega}^{T}\mathbf{v})(\mathbf{Z}\mathbf{v} + \mathbf{e})^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)]$$
$$[\mathbf{X}((\mathbf{X}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})^{-1})^{\boldsymbol{T}}(\boldsymbol{\lambda}^{T} - \boldsymbol{\omega}^{T}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}}\mathbf{V}^{-\mathbf{1}}(\delta)\mathbf{X})^{T}]$$
$$= 0.$$

Karena $cov\left[(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t}), \mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\delta) - \boldsymbol{\beta}\big)\right] = 0$, maka terbukti bahwa $(\hat{\boldsymbol{\tau}}^{*}(\boldsymbol{\delta},\boldsymbol{\beta},\mathbf{y}) - \boldsymbol{t})$ dan $\mathbf{d}^{\boldsymbol{T}}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\big)$ independen.

Lampiran 5. Bukti $\boldsymbol{\Upsilon}_{jk}(\boldsymbol{\delta}) = -E\left(\frac{\partial^2 \ln L(\boldsymbol{\beta},\boldsymbol{\delta};\boldsymbol{y})}{\partial\delta_k\partial\delta_j}\right) = \frac{1}{2}tr(\mathbf{V^{-1}}(\widehat{\boldsymbol{\delta}})\mathbf{V}_{(j)}\mathbf{V^{-1}}(\widehat{\boldsymbol{\delta}})\mathbf{V}_{(k)})$

Akan ditunjukkan bahwa:

$$\boldsymbol{\Upsilon}_{jk}(\boldsymbol{\delta}) = -E\left(\frac{\partial^2 \ln L(\boldsymbol{\beta},\boldsymbol{\delta};\boldsymbol{y})}{\partial\delta_k\partial\delta_j}\right) = \frac{1}{2}tr(\mathbf{V^{-1}}(\widehat{\boldsymbol{\delta}})\mathbf{V}_{(j)}\mathbf{V^{-1}}(\widehat{\boldsymbol{\delta}})\mathbf{V}_{(k)})$$

dengan $\boldsymbol{\Upsilon}_{jk}(\boldsymbol{\delta})$ elemen ke-jk dari $\boldsymbol{\Upsilon}(\boldsymbol{\delta})$, $\mathbf{V}_{(j)} = \frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_j}$, dan $\mathbf{V}_{(j,k)} = \frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_k\partial\delta_j}$,

sedemikian sehingga *Scoring Algorithm* dapat digunakan.

$$\boldsymbol{\Upsilon}_{jk}(\boldsymbol{\delta}) = -E\left(\frac{\partial^2 \ln L(\boldsymbol{\beta},\boldsymbol{\delta};\boldsymbol{y})}{\partial\delta_k\partial\delta_j}\right)$$

$$= -E\left(\frac{\partial^2\left(c - \frac{1}{2}(\ln(|\mathbf{V}(\boldsymbol{\delta})|) + (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T\mathbf{V^{-1}}(\boldsymbol{\delta})(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}))\right)}{\partial\delta_k\partial\delta_j}\right)$$

$$= -E\left(\frac{\partial^2 c}{\partial\delta_k\partial\delta_j} - \frac{1}{2}\frac{\partial^2\ln(|\mathbf{V}(\boldsymbol{\delta})|)}{\partial\delta_k\partial\delta_j} - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T\frac{\partial^2\mathbf{V^{-1}}(\boldsymbol{\delta})}{\partial\delta_k\partial\delta_j}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)$$

$$= -E\left(\begin{array}{c} 0 - \frac{1}{2}\left(tr\left(\mathbf{V}^{-1}(\boldsymbol{\delta})\frac{\partial^2\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_k\partial\delta_j}\right) - tr\left(\mathbf{V^{-1}}(\boldsymbol{\delta})\frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_k}\mathbf{V^{-1}}(\boldsymbol{\delta})\frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_j}\right)\right) - \frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^T \\ \left(-\mathbf{V^{-1}}(\boldsymbol{\delta})\frac{\partial^2\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_k\partial\delta_j}\mathbf{V^{-1}}(\boldsymbol{\delta}) + \mathbf{V}^{-1}(\boldsymbol{\delta})\frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_k}\mathbf{V}^{-1}(\boldsymbol{\delta})\frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_j}\mathbf{V^{-1}}(\boldsymbol{\delta}) + \mathbf{V^{-1}}(\boldsymbol{\delta})\frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_j}\mathbf{V}^{-1}(\boldsymbol{\delta})\frac{\partial\mathbf{V}(\boldsymbol{\delta})}{\partial\delta_k}\mathbf{V}^{-1}(\boldsymbol{\delta})\right)(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \end{array}\right)$$

Lampiran 5 (lanjutan)

$$= -E\begin{pmatrix} 0 - \frac{1}{2}\left(tr\left(\mathbf{V^{-1}(\delta)V}_{(j,k)}\right) - tr\left(\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\right)\right) - \frac{1}{2}(\mathbf{y}-\mathbf{X\beta})^T \\ \left(-\mathbf{V^{-1}(\delta)V}_{(j,k)}\mathbf{V^{-1}(\delta)} + \mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)} + \mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)}\right)(\mathbf{y}-\mathbf{X\beta}) \end{pmatrix}$$

$$= -E\begin{pmatrix} -\frac{1}{2}\left(tr\left(\mathbf{V^{-1}(\delta)V}_{(j,k)}\right) - tr\left(\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\right)\right) \\ -\frac{1}{2}tr((\mathbf{y}-\mathbf{X\beta})^T\begin{pmatrix} -\mathbf{V^{-1}(\delta)V}_{(j,k)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)} \end{pmatrix}(\mathbf{y}-\mathbf{X\beta})) \end{pmatrix}$$

$$= -E\begin{pmatrix} -\frac{1}{2}\left(tr\left(\mathbf{V^{-1}(\delta)V}_{(j,k)}\right) - tr\left(\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\right)\right) \\ -\frac{1}{2}tr\left(\begin{pmatrix} -\mathbf{V^{-1}(\delta)V}_{(j,k)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)} \end{pmatrix}(\mathbf{y}-\mathbf{X\beta})(\mathbf{y}-\mathbf{X\beta})^T\right) \end{pmatrix}$$

$$= -E\left(-\frac{1}{2}\left(tr\left(\mathbf{V^{-1}(\delta)V}_{(j,k)}\right) - tr\left(\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\right) + tr\left(\begin{pmatrix} -\mathbf{V^{-1}(\delta)V}_{(j,k)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)V}_{(j)}\mathbf{V^{-1}(\delta)V}_{(k)}\mathbf{V^{-1}(\delta)} \end{pmatrix}(\mathbf{y}-\mathbf{X\beta})(\mathbf{y}-\mathbf{X\beta})^T\right)\right)\right)$$

Lampiran 5 (lanjutan)

$$= -E\left(-\frac{1}{2}\left(tr\left(\left(\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)}\right) - \left(\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\right) + \begin{pmatrix} -\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)} \end{pmatrix}(\mathbf{y}-\mathbf{X\beta})(\mathbf{y}-\mathbf{X\beta})^T\right)\right)\right)$$

$$= \frac{1}{2}tr\left(\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)} - \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)} + \begin{pmatrix} -\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)}\mathbf{V^{-1}(\delta)} + \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)} \end{pmatrix} E\left((\mathbf{y}-\mathbf{X\beta})(\mathbf{y}-\mathbf{X\beta})^T\right)\right)$$

$$= \frac{1}{2}tr\left(\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)} - \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)} + \begin{pmatrix} -\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)}\mathbf{V^{-1}(\delta)} + \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)} \\ +\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)} \end{pmatrix}\mathbf{V(\delta)}\right)$$

$$= \frac{1}{2}tr\Big(\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)} - \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)} - \mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)}\mathbf{V^{-1}(\delta)}\mathbf{V(\delta)} + \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)}\mathbf{V(\delta)}$$

$$+ \mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V(\delta)}\Big)$$

$$= \frac{1}{2}tr\left(\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)} - \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)} - \mathbf{V^{-1}(\delta)}\mathbf{V}_{(j,k)} + \mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)} + \mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\right)$$

$$= \frac{1}{2}tr\left(\mathbf{V^{-1}(\delta)}\mathbf{V}_{(j)}\mathbf{V^{-1}(\delta)}\mathbf{V}_{(k)}\right) \qquad \text{(terbukti).}$$

Lampiran 6. Bukti $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ dan $\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})$ Independen

Akan ditunjukkan bahwa $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ dan $\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})$ independen, yaitu:

$$cov\left[(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}), \left(\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})\right)\right] = 0$$

dengan diberikan $\hat{\boldsymbol{\delta}}$ *translation-invariant* dan diasumsikan bahwa **v** dan **e** berdistribusi normal.

Bukti:

Berdasarkan subbab 2.6, diketahui bahwa jika distribusi bersama dari $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ dan $\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ adalah multivariat normal, maka $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ independen terhadap $\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ dan $\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}$ independen terhadap setiap fungsi dari $\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ (Kackar, 1984).

Kemudian, dengan diberikan $\hat{\boldsymbol{\delta}}$ *translation-invariant* dan diasumsikan bahwa **v** dan **e** berdistribusi normal, akan ditunjukkan terlebih dahulu bahwa $\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})$ merupakan fungsi dari $\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$.

$$\begin{aligned}\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) &= \left(\boldsymbol{\lambda}^T\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}) + \boldsymbol{\omega}^T\,\hat{\boldsymbol{\nu}}(\hat{\boldsymbol{\delta}})\right) - \left(\boldsymbol{\lambda}^T\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}) + \boldsymbol{\omega}^T\,\hat{\boldsymbol{\nu}}(\boldsymbol{\delta})\right)\\ &= \boldsymbol{\lambda}^T\underbrace{\left(\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)}_{\mathbf{1}} + \boldsymbol{\omega}^T\underbrace{\left(\hat{\boldsymbol{\nu}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\nu}}(\boldsymbol{\delta})\right)}_{\mathbf{2}}\end{aligned}$$

dimana bentuk 1 dan 2, sebagai berikut:

1. $\hat{\boldsymbol{\beta}}(\hat{\delta}) - \hat{\boldsymbol{\beta}}(\delta) = \left(\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{y} - (\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{y}$

$$\begin{aligned}&= \left(\mathbf{X}^T\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y} - \left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{y} + \hat{\boldsymbol{\beta}}(\delta) - \hat{\boldsymbol{\beta}}(\delta)\\
&= \left[\left(\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{y} - \hat{\boldsymbol{\beta}}(\delta)\right] - \left[\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{y} - \hat{\boldsymbol{\beta}}(\delta)\right]\\
&= \left[\left(\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{y} - \left(\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\hat{\delta})\mathbf{X}\hat{\boldsymbol{\beta}}(\delta)\right]\\
&\quad - \left[\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{y} - (\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\hat{\boldsymbol{\beta}}(\delta)\right]\\
&= \left[\left(\mathbf{X}^T\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\delta)\right)\right]\\
&\quad [\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}))]\\
&= \left[\left(\mathbf{X}^T\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}}) - \left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\right]\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right).\end{aligned}$$

Lampiran 6 (lanjutan)

2. $\hat{\boldsymbol{\nu}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\nu}}(\boldsymbol{\delta}) = \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}})\right) - \mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}))$

$$= \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y} - \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}) - \mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}))$$

$$= \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y} - \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y}$$
$$-\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}))$$

$$= \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y} - \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y}$$
$$-\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) + \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$$

$$= \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y} - \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$$
$$-\mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y} + \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$$
$$-\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)$$

$$= \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y} - \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$$
$$-\mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{y}$$
$$+\mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$$
$$-\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)$$

$$= \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)$$
$$-\mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)$$
$$-\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)$$

$$= \begin{bmatrix} \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}}) - \mathbf{G}(\hat{\boldsymbol{\delta}})\mathbf{Z}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\hat{\boldsymbol{\delta}}) \\ -\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta}) \end{bmatrix}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right).$$

Berdasarkan 1 dan 2, terlihat bahwa $\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ dan $\hat{\boldsymbol{\nu}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\nu}}(\boldsymbol{\delta})$ merupakan fungsi yang bergantung pada $\mathbf{y}$ yaitu hanya $\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$, sehingga $\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})$ merupakan fungsi yang *translation-invariant*.

Ambil $\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ adalah *maximal-invariant* dan setiap fungsi yang *translation-invariant* dari $\mathbf{y}$ dapat diekspresikan sebagai fungsi *maximal-invariant*. Berikut akan ditunjukkan bahwa $\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})$ dan $(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t})$ independen.

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$$cov\left[\left(\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})\right), (\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}))\right] = cov[\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}), (\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t})]$$

$$= E\left[(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})) - \underbrace{E[\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})]}_{i}\right]\left[(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}) - \underbrace{E[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}]}_{ii}\right]^T$$

Berikut perhitungan untuk i dan ii :

$$i.\ E[\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})] = E[\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})]$$

$$= E[\mathbf{X}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}(\boldsymbol{\delta})) + \mathbf{Z}\mathbf{v} + \mathbf{e}]$$

$$= E\left[\mathbf{X}.-\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Z}\mathbf{v} + \mathbf{e}) + \mathbf{Z}\mathbf{v} + \mathbf{e}\right]$$

$$= -\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})E[\mathbf{Z}\mathbf{v} + \mathbf{e}] + \mathbf{Z}E[\mathbf{v}] + E[\mathbf{e}]$$

$$= -\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta}).0 + \mathbf{Z}.0 + 0$$

$$= 0.$$

$$ii.\ E[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}] = E\left[\left(\boldsymbol{\lambda}^T\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}) + \boldsymbol{\omega}^T\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\right) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v})\right]$$

$$= E\left[\lambda^T\left(\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right) + \boldsymbol{\omega}^T\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) - \boldsymbol{\omega}^T\mathbf{v}\right]$$

$$= E\left[\boldsymbol{\lambda}^T\left(\hat{\boldsymbol{\beta}}(\delta) - \boldsymbol{\beta}\right) + \boldsymbol{\omega}^T\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{X}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}(\boldsymbol{\delta})) + \mathbf{Z}\mathbf{v} + \mathbf{e}\right) - \boldsymbol{\omega}^T\mathbf{v}\right]$$

$$= E\left[\begin{array}{c}\boldsymbol{\lambda}^T\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)(\mathbf{Z}\mathbf{v} + \mathbf{e}) \\ +\boldsymbol{\omega}^T\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{X}.-\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)(\mathbf{Z}\mathbf{v} + \mathbf{e}) + \mathbf{Z}\mathbf{v} + \mathbf{e}\right) - \left(\boldsymbol{\omega}^T\mathbf{v}\right)\end{array}\right]$$

$$= \boldsymbol{\lambda}^T\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)E[\mathbf{Z}\mathbf{v} + \mathbf{e}]$$

$$-\boldsymbol{\omega}^T\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)E[\mathbf{Z}\mathbf{v} + \mathbf{e}]$$

$$+\boldsymbol{\omega}^T\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})E[\mathbf{Z}\mathbf{v} + \mathbf{e}] - \boldsymbol{\omega}^TE[\mathbf{v}]$$

$$= \boldsymbol{\lambda}^T\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta}).0 - \boldsymbol{\omega}^T\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta}).0$$

$$+\boldsymbol{\omega}^T\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\delta).0 - \boldsymbol{\omega}^T.0$$

$$= 0.$$

Dari persamaan i dan ii, maka didapat:

$$cov\left[\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right), (\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t})\right] = E[\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})][(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t})]^T$$

$$= E\left[(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e}) - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right]$$

$$\left[\left(\boldsymbol{\lambda}^T\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}) + \boldsymbol{\omega}^T\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\delta)\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\right) - (\boldsymbol{\lambda}^T\boldsymbol{\beta} + \boldsymbol{\omega}^T\mathbf{v})\right]^T$$

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$$= E\left[\mathbf{X}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) + \mathbf{Zv} + \mathbf{e}\right]$$

$$\left[\boldsymbol{\lambda}^T\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right) + \boldsymbol{\omega}^T(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) - \mathbf{v})\right]^T$$

$$= E[\mathbf{X}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})) + \mathbf{Zv} + \mathbf{e}]\left[\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)^T\boldsymbol{\lambda} + (\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) - \mathbf{v})^T\boldsymbol{\omega}\right]$$

$$= E\begin{bmatrix} \mathbf{X}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)^T\boldsymbol{\lambda} + \mathbf{X}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) \\ \left(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) - \mathbf{v}\right)^T\boldsymbol{\omega} \\ +(\mathbf{Zv} + \mathbf{e})\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)^T\boldsymbol{\lambda} + (\mathbf{Zv} + \mathbf{e})(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) - \mathbf{v})^T\boldsymbol{\omega} \end{bmatrix}$$

$$= \underbrace{E\left[\mathbf{X}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)^T\boldsymbol{\lambda}\right]}_{I}$$

$$+ \underbrace{E\left[\mathbf{X}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\left(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) - \mathbf{v}\right)^T\boldsymbol{\omega}\right]}_{II}$$

$$+ \underbrace{E\left[(\mathbf{Zv} + \mathbf{e})\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)^T\boldsymbol{\lambda}\right]}_{III} + \underbrace{E\left[(\mathbf{Zv} + \mathbf{e})(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right) - \mathbf{v})^T\boldsymbol{\omega}\right]}_{IV}$$

Berikut akan diberikan penjabaran dari masing-masing nilai ekspektasi tersebut.

I. $E\left[\mathbf{X}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)^T\boldsymbol{\lambda}\right]$

$$= \mathbf{X}E\left[\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}\right)^T\right]\boldsymbol{\lambda}$$

$$= \mathbf{X}E\begin{bmatrix} \left(-\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv} + \mathbf{e})\right) \\ \left(\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv} + \mathbf{e})\right)^T \end{bmatrix}\boldsymbol{\lambda}$$

$$= \mathbf{X}E\left[-\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv} + \mathbf{e})(\mathbf{Zv} + \mathbf{e})^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\right]\boldsymbol{\lambda}$$

$$= -\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})E[(\mathbf{Zv} + \mathbf{e})(\mathbf{Zv} + \mathbf{e})^T]\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

$$= -\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{V}(\boldsymbol{\delta})\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

$$= -\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{I}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\boldsymbol{\lambda}$$

$$= -\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\underbrace{\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}}_{\mathbf{I}}\boldsymbol{\lambda}$$

$$= -\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\boldsymbol{\lambda}.$$

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$$II.E\left[\mathbf{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)\left(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)-\mathbf{v}\right)^{T}\boldsymbol{\omega}\right]$$

$$=E\begin{bmatrix}\mathbf{X}.-\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\\ \left(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{X}\boldsymbol{\beta}+\mathbf{Zv}+\mathbf{e}-\mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta})\right)-\mathbf{v}\right)^{T}\boldsymbol{\omega}\end{bmatrix}$$

$$=E\begin{bmatrix}-\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\\ \left(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{X}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}))+(\mathbf{Zv}+\mathbf{e})\right)-\mathbf{v}\right)^{T}\boldsymbol{\omega}\end{bmatrix}$$

$$=E\begin{bmatrix}-\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\\ \left(\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\left(\mathbf{X}.-\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})+(\mathbf{Zv}+\mathbf{e})\right)-\mathbf{v}\right)^{T}\boldsymbol{\omega}\end{bmatrix}$$

$$=E\begin{bmatrix}-\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\\ \left(\left(-\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})+(\mathbf{Zv}+\mathbf{e})\right)^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})-\mathbf{v}^{T}\right)\boldsymbol{\omega}\end{bmatrix}$$

$$=E\begin{bmatrix}-\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\\ \left(\left(-(\mathbf{Zv}+\mathbf{e})^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}+(\mathbf{Zv}+\mathbf{e})^{T}\right)\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})-\mathbf{v}^{T}\right)\boldsymbol{\omega}\end{bmatrix}$$

$$=E\begin{bmatrix}\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\\ (\mathbf{Zv}+\mathbf{e})^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}\\ -\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}\\ +\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\mathbf{v}^{T}\boldsymbol{\omega}\end{bmatrix}$$

$$=E\begin{bmatrix}\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\\ (\mathbf{Zv}+\mathbf{e})^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}\end{bmatrix}$$

$$-E\left[\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}\right]$$

$$+E\left[\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})(\mathbf{Zv}+\mathbf{e})\mathbf{v}^{T}\boldsymbol{\omega}\right]$$

$$=\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})E[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^{T}]$$

$$\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$-\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})E[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^{T}]\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$+\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})E[(\mathbf{Zv}+\mathbf{e})\mathbf{v}^{T}]\boldsymbol{\omega}$$

$$=\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{V}(\boldsymbol{\delta})\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$-\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{V}(\boldsymbol{\delta})\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$+\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

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$$= \mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega} - \mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}$$
$$+\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}$$
$$= \mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}$$

III. $E\left[(\mathbf{Zv}+\mathbf{e})\left(\widehat{\boldsymbol{\beta}}(\delta)-\boldsymbol{\beta}\right)^T\boldsymbol{\lambda}\right]$

$$= E\left[(\mathbf{Zv}+\mathbf{e})\left(\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)(\mathbf{Zv}+\mathbf{e})\right)^T\boldsymbol{\lambda}\right]$$
$$= E\left[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\lambda}\right]$$
$$= E[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T]\mathbf{V}^{-1}(\delta)\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\lambda}$$
$$= \mathbf{V}(\delta)\mathbf{V}^{-1}(\delta)\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\lambda}$$
$$= \mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\lambda}.$$

IV. $E\left[(\mathbf{Zv}+\mathbf{e})\left(\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\delta)\left(\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}}(\delta)\right)-\mathbf{v}\right)^T\boldsymbol{\omega}\right]$

$$= E[(\mathbf{Zv}+\mathbf{e})(\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\delta)\left(\mathbf{X}\boldsymbol{\beta}+\mathbf{Zv}+\mathbf{e}-\mathbf{X}\widehat{\boldsymbol{\beta}}(\delta)\right)-\mathbf{v})^T\boldsymbol{\omega}]$$
$$= E[(\mathbf{Zv}+\mathbf{e})(\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\delta)\left(\mathbf{X}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\delta))+(\mathbf{Zv}+\mathbf{e})\right)-\mathbf{v})^T\boldsymbol{\omega}]$$
$$= E[(\mathbf{Zv}+\mathbf{e})(\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\delta)\mathbf{X}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\delta))+\mathbf{G}(\delta)\mathbf{Z}^T\mathbf{V}^{-1}(\delta)(\mathbf{Zv}+\mathbf{e})-\mathbf{v})^T\boldsymbol{\omega}]$$
$$= E[(\mathbf{Zv}+\mathbf{e})\left((\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\delta))^T\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)+(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}(\delta)-\mathbf{v}^T\right)\boldsymbol{\omega}]$$
$$= E\begin{bmatrix}(\mathbf{Zv}+\mathbf{e})\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\delta)\right)^T\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}\\ +(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}(\delta)\boldsymbol{\omega}-(\mathbf{Zv}+\mathbf{e})\mathbf{v}^T\boldsymbol{\omega}\end{bmatrix}$$
$$= E\begin{bmatrix}(\mathbf{Zv}+\mathbf{e})\left(-\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)(\mathbf{Zv}+\mathbf{e})\right)^T\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}\\ +(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}(\delta)\boldsymbol{\omega}-(\mathbf{Zv}+\mathbf{e})\mathbf{v}^T\boldsymbol{\omega}\end{bmatrix}$$
$$= E\begin{bmatrix}-(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}\\ +(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}(\delta)\boldsymbol{\omega}-(\mathbf{Zv}+\mathbf{e})\mathbf{v}^T\boldsymbol{\omega}\end{bmatrix}$$
$$= -E\left[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}\right]$$
$$+E\left[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}(\delta)\boldsymbol{\omega}\right]-E[(\mathbf{Zv}+\mathbf{e})\mathbf{v}^T\boldsymbol{\omega}]$$
$$= -E[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T]\mathbf{V}^{-1}(\delta)\mathbf{X}\left(\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}^T(\delta)\boldsymbol{\omega}$$
$$+E[(\mathbf{Zv}+\mathbf{e})(\mathbf{Zv}+\mathbf{e})^T]\mathbf{V}^{-1}(\delta)\mathbf{Z}\mathbf{G}(\delta)\boldsymbol{\omega}-E[(\mathbf{Zv}+\mathbf{e})\mathbf{v}^T]\boldsymbol{\omega}$$

Lampiran 6 (lanjutan)

$$= -\underbrace{\mathbf{V}(\boldsymbol{\delta})\mathbf{V}^{-1}(\boldsymbol{\delta})}_{\mathbf{I}}\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$+\underbrace{\mathbf{V}(\boldsymbol{\delta})\mathbf{V}^{-1}(\boldsymbol{\delta})}_{\mathbf{I}}\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega} - \mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$= -\mathbf{X}\left(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}\right)^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}.$$

Berdasarkan I, II, III, dan IV diperoleh:

$$cov\left[\left(\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})\right), (\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}))\right]$$

$$= -\mathbf{X}(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\boldsymbol{\lambda} + \mathbf{X}(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$+\mathbf{X}(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\boldsymbol{\lambda} - \mathbf{X}(\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{Z}\mathbf{G}^{T}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$= 0.$$

Karena $cov\left[\left(\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})\right), (\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}))\right] = 0$, maka $\left(\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})\right)$ dan $(\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) - \boldsymbol{t}))$ independen, sehingga:

$$\text{MSE}\left[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})\right] = \text{MSE}[\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})] + E[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^{2}.$$

Lampiran 7. Bukti $E[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^2 \approx tr\left[\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}}\right)\mathbf{V}\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}}\right)^T I_\delta^{-1}(\boldsymbol{\delta})\right] =: g_3(\boldsymbol{\delta})$

Akan ditunjukkan bahwa:

$$E[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^2 \approx tr\left[\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}}\right)\mathbf{V}\left(\frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}}\right)^T I_\delta^{-1}(\boldsymbol{\delta})\right] =: g_3(\boldsymbol{\delta}).$$

Dengan menggunakan aproksimasi Taylor orde pertama dari $\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})$ di sekitar $\boldsymbol{\delta}$.

$$\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) \approx \hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) + \frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

$$\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta}) \approx \frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

$$[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^2 \approx \left[\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})\right]^2$$

$$\approx \left[\left(\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right)^T (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})\right]^2$$

Sebelumnya telah dijelaskan bahwa:

$$\hat{\boldsymbol{\tau}}(\boldsymbol{\delta}, \mathbf{y}) = \hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{y}) + \mathbf{d}^T(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta})$$

$$\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} = \frac{\partial(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta}, \boldsymbol{\beta}, \mathrm{y}) + \mathbf{d}^T(\widehat{\boldsymbol{\beta}}(\boldsymbol{\delta}) - \boldsymbol{\beta}))}{\partial \boldsymbol{\delta}}$$

Dengan asumsi $\frac{\partial \mathbf{d}^T}{\partial \delta} \approx 0$, diperoleh:

$$\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} \approx \frac{\partial(\hat{\boldsymbol{\tau}}^*(\boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{y}))}{\partial \boldsymbol{\delta}}$$

$$\approx \frac{\partial(\boldsymbol{\lambda}^T \boldsymbol{\beta} + \mathbf{a}^T(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}))}{\partial \boldsymbol{\delta}}$$

$$\approx \frac{\partial(\mathbf{a}^T(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}))}{\partial \boldsymbol{\delta}}$$

$$\approx \frac{\partial \mathbf{a}^T}{\partial \boldsymbol{\delta}}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$$

maka:

$$[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}}) - \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^2 \approx \left[\left(\frac{\partial \hat{\boldsymbol{\tau}}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right)^T (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})\right]^2$$

Lampiran 7 (lanjutan)

$$\approx \left[\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)^{T}(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta})\right]^{2}$$

sehingga dengan mengambil nilai ekspektasinya, didapat:

$$E[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})-\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^{2} \approx E\left[\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)^{T}(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta})\right]^{2}$$

$$\approx tr\left[E\left(\left(\frac{\partial \mathrm{a}^{\boldsymbol{T}}}{\partial \delta}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)\left(\frac{\partial \mathrm{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)^{T}\right).\underbrace{E((\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta})^{T})}_{I_{\delta}^{-1}(\boldsymbol{\delta})}\right]$$

$$\approx tr\left[E\left(\frac{\partial \mathrm{a}^{\boldsymbol{T}}}{\partial \delta}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^{T}\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}\right)^{T}\right).I_{\delta}^{-1}(\widehat{\boldsymbol{\delta}})\right]$$

$$\approx tr\left[\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}E((\mathbf{y}-\mathbf{X}\boldsymbol{\beta})(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^{\boldsymbol{T}})\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}\right)^{T}.I_{\delta}^{-1}(\widehat{\boldsymbol{\delta}})\right]$$

$$\approx tr\left[\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}.\mathbf{V}(\boldsymbol{\delta}).\left(\frac{\partial \mathrm{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}}\right)^{T}.I_{\delta}^{-1}(\widehat{\boldsymbol{\delta}})\right]$$

$$=: g_{3}(\boldsymbol{\delta}).$$

dimana

$$I_{\delta}^{-1}(\widehat{\boldsymbol{\delta}}) = var(\widehat{\boldsymbol{\delta}})$$

$$= \left[-E\left(\frac{\partial^{2}\ln L(\boldsymbol{\beta},\boldsymbol{\delta};\boldsymbol{y})}{\partial\delta_{k}\partial\delta_{j}}\right)\right]^{-1}$$, berdasarkan lampiran 5

$$= \left[\frac{1}{2}tr(\mathbf{V}^{-1}(\widehat{\boldsymbol{\delta}})\mathbf{V}_{(j)}\mathbf{V}^{-1}(\widehat{\boldsymbol{\delta}})\mathbf{V}_{(k)})\right]^{-1}$$

Dengan demikian, $E[\hat{\boldsymbol{\tau}}(\widehat{\boldsymbol{\delta}})-\hat{\boldsymbol{\tau}}(\boldsymbol{\delta})]^{2} \approx g_{3}(\boldsymbol{\delta})$ (terbukti).

Lampiran 8. Bukti $E\left[g_1(\hat{\boldsymbol{\delta}})\right] \approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$,
$E\left[g_2(\hat{\boldsymbol{\delta}})\right] \approx g_2(\boldsymbol{\delta})$, dan $E\left[g_3(\hat{\boldsymbol{\delta}})\right] \approx g_3(\boldsymbol{\delta})$.

a. Akan ditunjukkan bahwa $E\left[g_1(\hat{\boldsymbol{\delta}})\right] \approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$.

Dengan menggunakan ekspansi Taylor untuk $g_1(\boldsymbol{\delta})$di sekitar $\boldsymbol{\delta}$, yaitu:

$$g_1(\hat{\boldsymbol{\delta}}) = g_1(\boldsymbol{\delta}) + (\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_1(\boldsymbol{\delta}) + \frac{1}{2}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_1(\boldsymbol{\delta})(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}) + r(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

dimana

$\nabla g_1(\boldsymbol{\delta})$ adalah vektor dari turunan pertama $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$.

$\nabla^2 g_1(\boldsymbol{\delta})$adalah matriks turunan kedua $g_1(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$.

$\lim_{\hat{\delta}\to\delta} \frac{r(\hat{\delta}-\delta)}{|\hat{\delta}-\delta|^2} = 0$, sehingga

$$g_1(\hat{\boldsymbol{\delta}}) = g_1(\boldsymbol{\delta}) + (\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_1(\boldsymbol{\delta}) + \frac{1}{2}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_1(\boldsymbol{\delta})(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

Diketahui

$$\begin{aligned} g_1(\boldsymbol{\delta}) &= \boldsymbol{\omega}^T \left(\mathbf{G}(\boldsymbol{\delta}) - \mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\right)\boldsymbol{\omega} \\ &= \boldsymbol{\omega}^T\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega} - \underbrace{\boldsymbol{\omega}^T\mathbf{G}(\boldsymbol{\delta})\mathbf{Z}^{\mathbf{T}}\mathbf{V}^{-\mathbf{1}}}_{\mathbf{a}^T}\mathbf{Z}\mathbf{G}\boldsymbol{\omega} \\ &= \boldsymbol{\omega}^T\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega} - \mathbf{a}^T\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}\text{ , maka:} \end{aligned}$$

$$\frac{\partial g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_d} = \boldsymbol{\omega}^T \frac{\partial \mathbf{G}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_d}\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega} - \mathbf{a}^T \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$\begin{aligned} \frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} &= \boldsymbol{\omega}^T \frac{\partial \mathbf{G}(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e\, \partial \boldsymbol{\delta}_d}\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e\, \partial \boldsymbol{\delta}_d}\right)^T \mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega} - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}) \\ &\quad - \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}) - \mathbf{a}^T \frac{\partial}{\partial \boldsymbol{\delta}_e\, \partial \boldsymbol{\delta}_d}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}) \end{aligned}$$

$$= \underbrace{-\left(\frac{\partial^2 \mathbf{a}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\right)^T \mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}}_{i} - \underbrace{\left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega})}_{ii} - \underbrace{\left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_e}\right)^T \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega})}_{iii}$$

dengan

$$\begin{aligned} \frac{\partial^2 \mathbf{a}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} &= \left(-\mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \frac{\partial \mathbf{V}^{-\mathbf{1}}}{\partial \boldsymbol{\delta}_e} - \mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\mathbf{V}^{-\mathbf{1}} - \frac{\partial \mathbf{V}^{-\mathbf{1}}}{\partial \boldsymbol{\delta}_e} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-\mathbf{1}}\right)\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega} \\ &\quad -\mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}) + \frac{\partial \mathbf{V}^{-\mathbf{1}}}{\partial \boldsymbol{\delta}_e} \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}) \\ &\quad +\mathbf{V}^{-\mathbf{1}} \frac{\partial^2}{\partial \boldsymbol{\delta}_e\, \partial \boldsymbol{\delta}_d}(\mathbf{Z}\mathbf{G}(\boldsymbol{\delta})\boldsymbol{\omega}) \end{aligned}$$

Lampiran 8 (lanjutan)

$$= \left(\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1} + \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\right)\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$-\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) - \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$= \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\underbrace{\mathbf{V}^{-1}\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}}_{\mathbf{a}} + \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\underbrace{\mathbf{V}^{-1}\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}}_{\mathbf{a}}$$

$$-\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) - \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$= \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{a} - \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$-\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$\frac{\partial \mathbf{a}}{\partial \boldsymbol{\delta}_d} = \frac{\partial \mathbf{V}^{-1}}{\partial \boldsymbol{\delta}_d}\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} + \mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$= -\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega} + \mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$= -\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{a} + \mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

Selanjutnya $(i), (ii),$ dan (iii) dijabarkan sebagai berikut:

$(i) - \left(\frac{\partial^2 \mathbf{a}}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}\right)^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}$

$$= -\begin{bmatrix} \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{a} - \mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \\ -\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \end{bmatrix}^T \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$= -\begin{bmatrix} \mathbf{a}^T\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1} + \mathbf{a}^T\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1} - \frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1} \\ -\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1} \end{bmatrix} \mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}$$

$$= -\begin{bmatrix} \mathbf{a}^T\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{a} + \mathbf{a}^T\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} - \frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{a} \\ -\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} \end{bmatrix}$$

Lampiran 8 (lanjutan)

$$= \begin{bmatrix} -\mathbf{a}^{\boldsymbol{T}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-\mathbf{1}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} - \mathbf{a}^{\boldsymbol{T}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{V}^{-\mathbf{1}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} + \dfrac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} \\ + \dfrac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbf{a}^{\boldsymbol{T}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-\mathbf{1}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} - \mathbf{a}^{\boldsymbol{T}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-\mathbf{1}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} + \dfrac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} \\ + \dfrac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\delta)\omega)^T \mathbf{V}^{-1} \dfrac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} \end{bmatrix}$$

$$= \left[-2\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} + \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} + \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\delta)\omega)^T \mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} \right]$$

$(ii) - \left(\frac{\partial \mathbf{a}}{\partial \delta_d}\right)^T \frac{\partial}{\partial \delta_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$

$$= -\left[-\mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} + \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \right]^T \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\delta)\omega)$$

$$= -\left[-\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{V}^{-\mathbf{1}} + \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\delta)\omega)^T \mathbf{V}^{-\mathbf{1}} \right] \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\delta)\omega)$$

$$= \left[\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\delta)\boldsymbol{\omega}) - \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\delta)\boldsymbol{\omega})^T \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\delta)\omega) \right]$$

$(iii) - \left(\frac{\partial \mathbf{a}}{\partial \delta_e}\right)^T \frac{\partial}{\partial \delta_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$

$$= -\left[-\mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} + \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \right]^T \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\delta)\omega)$$

$$= -\left[-\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-\mathbf{1}} + \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\delta)\omega)^T \mathbf{V}^{-\mathbf{1}} \right] \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$= \left[\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) - \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\delta)\boldsymbol{\omega})^T \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\delta)\omega) \right]$$

$$= \left[\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) - \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\delta)\boldsymbol{\omega})^T \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \right]$$

Berdasarkan $(i), (ii),$ dan (iii) maka diperoleh:

Lampiran 8 (lanjutan)

$$\frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d} = \begin{bmatrix} -2\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} + \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} \\ + \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \\ - \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \\ - \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-1} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \end{bmatrix}$$

$$= -2\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} + 2\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

$$+2\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) - 2 \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})$$

Selanjutnya akan dicari nilai ekspektasi dari $g_1(\widehat{\boldsymbol{\delta}})$, yaitu:

$$\begin{aligned} E\left[g_1(\widehat{\boldsymbol{\delta}})\right] &\approx E\left[g_1(\boldsymbol{\delta}) + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_1(\boldsymbol{\delta}) + \tfrac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_1(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})\right] \\ &\approx E[g_1(\boldsymbol{\delta})] + E\left[(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_1(\boldsymbol{\delta})\right] + E\left[\tfrac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_1(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})\right] \\ &\approx E[g_1(\boldsymbol{\delta})] + E\left[(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_1(\boldsymbol{\delta})\right] + E\left[\tfrac{1}{2}\nabla^2 g_1(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T\right] \\ &\approx g_1(\boldsymbol{\delta}) + E\left[(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T\right] \nabla g_1(\boldsymbol{\delta}) + E\left[\tfrac{1}{2}\nabla^2 g_1(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T\right] \\ &\approx g_1(\boldsymbol{\delta}) + c_{\hat{\delta}}^T(\boldsymbol{\delta}) \nabla g_1(\boldsymbol{\delta}) + \tfrac{1}{2} tr[\nabla^2 g_1(\boldsymbol{\delta}) I_\delta^{-1}(\boldsymbol{\delta})] \end{aligned}$$

Berdasarkan lampiran 6, diketahui bahwa $g_3(\boldsymbol{\delta}) \approx: tr\left[\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \delta}\right) \mathbf{V} \left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \delta}\right)^T I_\delta^{-1}(\boldsymbol{\delta})\right]$.

Bentuk $tr\left[\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}_d}\right) \mathbf{V} \left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}_e}\right)^T I_\delta^{-1}(\boldsymbol{\delta})\right]$ ini dapat dituliskan dalam bentuk:

$$tr\left[\left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}_d}\right) \mathbf{V} \left(\frac{\partial \mathbf{a}^{\boldsymbol{T}}}{\partial \boldsymbol{\delta}_e}\right)^T I_\delta^{-1}(\boldsymbol{\delta})\right] = tr\begin{bmatrix} \left(-\mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{a} + \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})\right)^T \mathbf{V} \\ \left(-\mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} + \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})\right) I_\delta^{-1}(\boldsymbol{\delta}) \end{bmatrix}$$

$$= tr\begin{bmatrix} \left(-\mathbf{a}^{\boldsymbol{T}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d} \mathbf{V}^{-\mathbf{1}} + \frac{\partial}{\partial \boldsymbol{\delta}_d} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^T \mathbf{V}^{-\mathbf{1}}\right) \mathbf{V} \\ \left(-\mathbf{V}^{-\mathbf{1}} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e} \mathbf{a} + \mathbf{V}^{-\mathbf{1}} \frac{\partial}{\partial \boldsymbol{\delta}_e} (\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})\right) I_\delta^{-1}(\boldsymbol{\delta}) \end{bmatrix}$$

Lampiran 8 (lanjutan)

$$= tr\left[\left(-\mathbf{a}^{T}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\mathbf{V} + \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^{T}\mathbf{V}^{-1}\mathbf{V}\right)\left(-\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})\right)I_{\delta}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\left(-\mathbf{a}^{T}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{I} + \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^{T}\mathbf{I}\right)\left(-\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} + \mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})\right)I_{\delta}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\begin{pmatrix}\mathbf{a}^{T}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} - \mathbf{a}^{T}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) - \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^{T}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} \\ + \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^{T}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})\end{pmatrix}I_{\delta}^{-1}(\boldsymbol{\delta})\right]$$

$$= tr\left[\begin{pmatrix}\mathbf{a}^{T}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{a} - \mathbf{a}^{T}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_d}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) - \mathbf{a}^{T}\frac{\partial \mathbf{V}}{\partial \boldsymbol{\delta}_e}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega}) \\ + \frac{\partial}{\partial \boldsymbol{\delta}_d}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})^{T}\mathbf{V}^{-1}\frac{\partial}{\partial \boldsymbol{\delta}_e}(\mathbf{ZG}(\boldsymbol{\delta})\boldsymbol{\omega})\end{pmatrix}I_{\delta}^{-1}(\boldsymbol{\delta})\right]$$

$$= -\frac{1}{2}tr\left[\frac{\partial^2 g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_e \partial \boldsymbol{\delta}_d}I_{\delta}^{-1}(\boldsymbol{\delta})\right]$$

Karena $tr\left[\left(\frac{\partial \mathbf{a}^{T}}{\partial \delta_d}\right)\mathbf{V}\left(\frac{\partial \mathbf{a}^{T}}{\partial \delta_e}\right)^{T} I_{\delta}^{-1}(\boldsymbol{\delta})\right] = -\frac{1}{2}tr\left[\frac{\partial^2 g_1(\delta)}{\partial \delta_e \partial \delta_d}I_{\delta}^{-1}(\boldsymbol{\delta})\right]$, maka

$g_3(\boldsymbol{\delta}) = -\frac{1}{2}tr[\nabla^2 g_1(\boldsymbol{\delta})I_{\delta}^{-1}(\boldsymbol{\delta})]$, sehingga

$E\left[g_1(\widehat{\boldsymbol{\delta}})\right] \approx g_1(\boldsymbol{\delta}) + c_{\widehat{\delta}}^{T}(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$.

Jadi, $E\left[g_1(\widehat{\boldsymbol{\delta}})\right] \approx g_1(\boldsymbol{\delta}) + c_{\widehat{\delta}}^{T}(\boldsymbol{\delta})\nabla g_1(\boldsymbol{\delta}) - g_3(\boldsymbol{\delta})$.

b. Akan ditunjukkan bahwa $E\left[g_2(\widehat{\boldsymbol{\delta}})\right] \approx g_2(\boldsymbol{\delta})$.

Dengan menggunakan ekspansi Taylor untuk $g_2(\boldsymbol{\delta})$di sekitar $\boldsymbol{\delta}$, yaitu:

$$g_2(\widehat{\boldsymbol{\delta}}) = g_2(\boldsymbol{\delta}) + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^{T}\nabla g_2(\boldsymbol{\delta}) + \frac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^{T}\nabla^2 g_2(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}) + r(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

dimana

$\nabla g_2(\boldsymbol{\delta})$ adalah vektor dari turunan pertama $g_2(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$.

$\nabla^2 g_2(\boldsymbol{\delta})$adalah matriks turunan kedua $g_2(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$.

$\lim_{\widehat{\delta}\to\delta}\frac{r(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta})}{|\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta}|^2} = 0$, sehingga

Lampiran 8 (lanjutan)

$$g_2(\widehat{\boldsymbol{\delta}}) = g_2(\boldsymbol{\delta}) + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_2(\boldsymbol{\delta}) + \frac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_2(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

Diketahui $g_2(\boldsymbol{\delta}) = \mathbf{d}^T(\mathbf{X}^T\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X})^{-1}\mathbf{d}$, dengan $\mathbf{d}^T = \boldsymbol{\lambda}^T - \boldsymbol{\omega}^T\mathbf{G}\mathbf{Z}^{\mathbf{T}}\mathbf{V}^{-1}(\boldsymbol{\delta})\mathbf{X}$.

Dengan asumsi $\frac{\partial \mathbf{d}^T}{\partial \delta} \approx 0$, maka$\nabla g_2(\boldsymbol{\delta}) \approx 0$, sehingga diperoleh:

$$E[g_2(\widehat{\boldsymbol{\delta}})] \approx E[g_2(\boldsymbol{\delta}) + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_2(\boldsymbol{\delta}) + \frac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_2(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})]$$

$$\approx g_2(\boldsymbol{\delta})$$

Jadi, $E[g_2(\widehat{\boldsymbol{\delta}})] \approx g_2(\boldsymbol{\delta})$.

c. Akan ditunjukkan bahwa $E\left[g_3(\widehat{\boldsymbol{\delta}})\right] \approx g_3(\boldsymbol{\delta})$.

Dengan menggunakan ekspansi Taylor untuk $g_3(\boldsymbol{\delta})$ di sekitar $\boldsymbol{\delta}$, yaitu:

$$g_3(\widehat{\boldsymbol{\delta}}) = g_3(\boldsymbol{\delta}) + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_3(\boldsymbol{\delta}) + \frac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_3(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}) + r(\widehat{\boldsymbol{\delta}} - \delta)$$

dimana:

$\nabla g_3(\boldsymbol{\delta})$ adalah vektor dari turunan pertama $g_3(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$.

$\nabla^2 g_3(\boldsymbol{\delta})$adalah matriks turunan kedua $g_3(\boldsymbol{\delta})$ terhadap $\boldsymbol{\delta}$.

$\lim_{\widehat{\delta} \to \delta} \frac{r(\widehat{\delta} - \delta)}{|\widehat{\boldsymbol{\delta}} - \delta|^2} = 0$, sehingga

$$g_3(\widehat{\boldsymbol{\delta}}) \approx g_3(\boldsymbol{\delta}) + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_3(\boldsymbol{\delta}) + \frac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_3(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

Diketahui $g_3(\boldsymbol{\delta}) = -\frac{1}{2} tr[\nabla^2 g_1(\boldsymbol{\delta}) I_\delta^{-1}(\delta)]$

sehingga:

$$g_3(\widehat{\boldsymbol{\delta}}) \approx g_3(\boldsymbol{\delta}) + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla g_3(\boldsymbol{\delta}) + \frac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \nabla^2 g_3(\boldsymbol{\delta})(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

$$\approx -\frac{1}{2} tr[\nabla^2 g_1(\boldsymbol{\delta}) I_\delta^{-1}(\delta)] + (\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \left(-\frac{1}{2} tr[\nabla^3 g_1(\boldsymbol{\delta}) I_\delta^{-1}(\delta)]\right) +$$

$$\frac{1}{2}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})^T \left(-\frac{1}{2} tr[\nabla^4 g_1(\boldsymbol{\delta}) I_\delta^{-1}(\delta)]\right)(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta})$$

$$\approx -\frac{1}{2} tr[\nabla^2 g_1(\boldsymbol{\delta}) I_\delta^{-1}(\delta)]$$

$$\approx g_3(\boldsymbol{\delta}).$$

Jadi, $E[g_3(\widehat{\boldsymbol{\delta}})] \approx g_3(\widehat{\boldsymbol{\delta}})$ (terbukti).

Lampiran 9. Metode BLUP pada Model Fay-Herriot

Pada metode BLUP, diasumsikan bahwa variansi pengaruh acak $(\sigma_{\boldsymbol{v}}^{\mathbf{2}})$ diketahui. Dalam hal ini, dimisalkan yang akan ditaksir pada model Fay-Herriot adalah kombinasi linear $\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i$, dengan $\boldsymbol{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{pmatrix}$, $\boldsymbol{\beta}$ berukuran $px1$, dan v_i berukuran $1x1$. Misalkan $T(\hat{\theta}_i) = l_i{}^T \hat{\theta}_i + c$ adalah sembarang penaksir kombinasi linear $\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i$, dimana l_i merupakan elemen dari vektor $\boldsymbol{l} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix}$ dengan $i = 1, 2, , n$ dan c suatu konstanta.

Selanjutnya, akan ditunjukkan bahwa sembarang penaksir $T(\hat{\theta}_i) = l_i{}^T \hat{\theta}_i + c$ tersebut merupakan penaksir yang *best*, linear, dan *unbiased*. Berikut akan diuraikan masing-masing sifat tersebut.

1. Linear

   Akan ditunjukkan bahwa penaksir $T(\hat{\theta}_i) = l_i{}^T \hat{\theta}_i + c$ bersifat linear. $T(\hat{\theta}_i)$ dikatakan penaksir yang linear, jika $T(p\hat{\theta}_{i1} + q\hat{\theta}_{i2}) = p\hat{\theta}_{i1} + q\hat{\theta}_{i2}$, sehingga:

$$
\begin{aligned}
T(p\hat{\theta}_{i1} + q\hat{\theta}_{i2}) &= l_i{}^T(p\hat{\theta}_{i1} + q\hat{\theta}_{i2}) + c \\
&= l_i{}^T p\hat{\theta}_{i1} + l_i q\hat{\theta}_{i2} + pc_1 + qc_2;\ \text{dengan } c_1 = \frac{c}{2p} \text{ dan } c_2 = \frac{c}{2q} \\
&= p(l_i{}^T\hat{\theta}_{i1}) + pc_1 + q(l_i{}^T\hat{\theta}_{i2}) + qc_2 \\
&= p(l_i{}^T\hat{\theta}_{i1} + c_1) + q(l_i{}^T\hat{\theta}_{i2} + c_2) \\
&= pT(\hat{\theta}_{i1}) + qT(\hat{\theta}_{i2}) \qquad \text{(i)}
\end{aligned}
$$

2. *Unbiased*

   Akan ditunjukkan bahwa $T(\hat{\theta}_i)$ bersifat *unbiased*. $T(\hat{\theta}_i)$ dikatakan bersifat *unbiased* jika $E(T(\hat{\theta}_i)) = E(\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i)$.

   Bukti:

Lampiran 9 (lanjutan)

Diketahui : $T(\hat{\theta}_i) = l_i{}^T\hat{\theta}_i + c$

$$E\left(T(\hat{\theta}_i)\right) = E\left(l_i{}^T\hat{\theta}_i + c\right)$$
$$= E\left(l_i{}^T\left(\mathbf{z_i^T}\boldsymbol{\beta} + v_i + e_i\right)\right) + E(c)$$
$$= E\left(l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta} + l_i{}^T v_i + l_i{}^T e_i\right) + E(c)$$
$$= E\left(l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta}\right) + E\left(l_i{}^T v_i\right) + E\left(l_i{}^T e_i\right) + c$$
$$= l_i{}^T E\left(\mathbf{z_i^T}\boldsymbol{\beta}\right) + l_i{}^T E(v_i) + l_i{}^T E(e_i) + c$$
$$= l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta} + l_i{}^T(0) + l_i{}^T(0) + c \text{ ; karena } E(v_i) = 0 \text{ dan } E(e_i) = 0$$
$$= l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta} + c \qquad \text{(ii)}$$

$$E(\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i) = E(\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta}) + E(v_i)$$
$$= \boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + 0 \text{ ; karena } E(v_i) = 0$$
$$= \boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} \qquad \text{(iii)}$$

Berdasarkan persamaan (ii) dan (iii), terlihat bahwa $T(\hat{\theta}_i)$ bersifat *biased* karena $E(T(\hat{\theta}_i)) \neq E(\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i)$. Berikut akan ditunjukkan bahwa jika $c = 0$ dan $\boldsymbol{\gamma}^{\boldsymbol{T}}$ kombinasi linear dari baris-baris $\mathbf{z_i^T}$, yaitu $\boldsymbol{\gamma}^{\boldsymbol{T}} = l_i{}^T\mathbf{z_i^T}$, maka $E\left(T(\hat{\theta}_i)\right) = E(\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i)$.

$$E\left(T(\hat{\theta}_i)\right) = E\left(l_i{}^T\hat{\theta}_i + c\right)$$
$$= E\left(l_i{}^T\left(\mathbf{z_i^T}\boldsymbol{\beta} + v_i + e_i\right) + 0\right) \text{ ; karena } c = 0.$$
$$= E\left(l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta} + l_i{}^T v_i + l_i{}^T e_i\right)$$
$$= E\left(l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta}\right) + E\left(l_i{}^T v_i\right) + E\left(l_i{}^T e_i\right)$$
$$= l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta} + l_i{}^T E(v_i) + l_i{}^T E(e_i)$$
$$= l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta} + l_i{}^T(0) + l_i{}^T(0) \text{ ; karena } E(v_i) = 0 \text{ dan } E(e_i) = 0$$
$$= l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta}$$
$$= \boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} \text{ ; karena } \boldsymbol{\gamma}^{\boldsymbol{T}} = l_i{}^T\mathbf{z_i^T} \qquad \text{(iv)}$$

Lampiran 9 (lanjutan)

Dengan demikian, jika $c = 0$ dan $\boldsymbol{\gamma}^{\boldsymbol{T}}$ kombinasi linear dari baris-baris $\mathbf{z_i^T}$, yaitu $\boldsymbol{\gamma}^{\boldsymbol{T}} = l_i{}^T \mathbf{z_i^T}$, maka $T(\hat{\theta}_i)$ bersifat *unbiased* terhadap $\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i$. Selanjutnya, $T(\hat{\theta}_i) = l_i{}^T \hat{\theta}_i$ ini akan digunakan sebagai penaksir untuk $\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i$.

3. Terbaik (*best*)

Penaksir $T(\hat{\theta}_i)$ dikatakan bersifat *best* jika $T(\hat{\theta}_i)$ memiliki MSE terkecil diantara penaksir *unbiased* dan linear lainnya (Rao, 2003). Definisikan $E[(T(\hat{\theta}_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i))^2]$ sebagai MSE dari $T(\hat{\theta}_i)$.

Dalam tugas akhir ini, akan ditunjukkan bahwa $E[(T(\hat{\theta}_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i))^2]$ minimum terhadap kendala $l_i{}^T \mathbf{z_i^T} - \boldsymbol{\gamma}^{\boldsymbol{T}} = \mathbf{0}$. Kemudian akan digunakan metode *Lagrange* untuk meminimumkan MSE dari $T(\hat{\theta}_i)$ terhadap satu kendala. Namun, sebelumnya akan didefinisikan fungsi-fungsi berikut:

$$\begin{aligned}
T(\hat{\theta}_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i) &= l_i{}^T(\mathbf{z_i^T}\boldsymbol{\beta} + v_i + e_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i) \\
&= l_i{}^T\mathbf{z_i^T}\boldsymbol{\beta} + l_i{}^T v_i + l_i{}^T e_i - \boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} - v_i \\
&= (l_i{}^T\mathbf{z_i^T} - \boldsymbol{\gamma}^{\boldsymbol{T}})\boldsymbol{\beta} + l_i{}^T(v_i + e_i) - v_i \\
&= (\mathbf{0})\boldsymbol{\beta} + l_i{}^T(v_i + e_i) - v_i \text{ ; karena } l_i{}^T\mathbf{z_i^T} - \boldsymbol{\gamma}^{\boldsymbol{T}} = \mathbf{0} \\
&= l_i{}^T(v_i + e_i) - v_i \qquad \text{(v)}
\end{aligned}$$

$$\begin{aligned}
(l_i{}^T(v_i + e_i) - v_i)^T &= (v_i + e_i)^T l_i - v_i{}^T \\
&= (v_i{}^T + e_i{}^T) l_i - v_i{}^T \qquad \text{(vi)}
\end{aligned}$$

MSE nya diperoleh dengan:

$$E[(T(\hat{\theta}_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i))^2] = E[(T(\hat{\theta}_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i))(T(\hat{\theta}_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i))^T]$$

$$= E[(l_i{}^T(v_i + e_i) - v_i)((v_i{}^T + e_i{}^T) l_i - v_i{}^T)]$$

$$= E[l_i{}^T(v_i + e_i)(v_i{}^T + e_i{}^T) l_i - l_i{}^T(v_i + e_i) v_i{}^T - v_i(v_i{}^T + e_i{}^T) l_i + v_i v_i{}^T]$$

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$$= \underbrace{E\left[l_i{}^T(v_i + e_i)(v_i{}^T + e_i{}^T)l_i\right]}_{1} - \underbrace{E\left[l_i{}^T(v_i + e_i)v_i{}^T\right]}_{2} - \underbrace{E[v_i(v_i{}^T + e_i{}^T)l_i]}_{3}$$
$$+ \underbrace{E[v_i v_i{}^T]}_{4}$$

Berikut akan dijelaskan lebih lanjut.

(1)$E\left[l_i{}^T(v_i + e_i)(v_i{}^T + e_i{}^T)l_i\right]$

$= l_i{}^T E[(v_i + e_i)(v_i{}^T + e_i{}^T)]l_i$

$= l_i{}^T E[v_i v_i{}^T + v_i e_i{}^T + e_i v_i{}^T + e_i e_i{}^T]l_i$

$= l_i{}^T \{E[v_i v_i{}^T] + E[v_i e_i{}^T] + E[e_i v_i{}^T] + E[e_i e_i{}^T]\}l_i$

$= l_i{}^T(\sigma_v^2 + 0 + 0 + \psi_i)l_i$; karena $E[v_i v_i{}^T] = \sigma_v^2$ dan $E[e_i e_i{}^T] = 0$

$= l_i{}^T(\sigma_v^2 + \psi_i)l_i$

(2)$E[l_i{}^T(v_i + e_i)v_i{}^T]$

$= l_i{}^T E[(v_i + e_i)v_i{}^T]$

$= l_i{}^T E[v_i v_i{}^T + e_i v_i{}^T]$

$= l_i{}^T \{E[v_i v_i{}^T] + E[e_i v_i{}^T]\}$

$= l_i{}^T(\sigma_v^2 + 0)$; karena $E[v_i v_i{}^T] = \sigma_v^2$ dan $E[e_i v_i{}^T] = 0$

$= l_i{}^T \sigma_v^2$

(3)$E[v_i(v_i{}^T + e_i{}^T)l_i]$

$= E[v_i(v_i{}^T + e_i{}^T)]l_i$

$= E[v_i v_i{}^T + v_i e_i{}^T]l_i$

$= \{E[v_i v_i{}^T] + E[v_i e_i{}^T]\}l_i$

$= (\sigma_v^2 + 0)l_i$; karena $E[v_i v_i{}^T] = \sigma_v^2$ dan $E[v_i e_i{}^T] = 0$

$= \sigma_v^2 l_i$

(4)$E[v_i v_i{}^T] = \sigma_v^2$

dari $(1), (2), \ldots, (4)$ diperoleh:

$$E\left[(T(\hat{\theta}_i) - (\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{\beta} + v_i))^2\right] = l_i{}^T(\sigma_v^2 + \psi_i)l_i - l_i{}^T\sigma_v^2 - \sigma_v^2 l_i + \sigma_v^2$$

karena $(\sigma_v^2 l_i)^T = l_i{}^T\sigma_v^2$ dan $(\sigma_v^2)^T = \sigma_v^2$, maka

$$= l_i{}^T(\sigma_v^2 + \psi_i)l_i - 2l_i{}^T\sigma_v^2 + \sigma_v^2 \qquad \text{(vii)}$$

Lampiran 9 (lanjutan)

Untuk mendapatkan taksiran terbaik, maka berdasarkan metode *Lagrange*, diperoleh fungsi *Lagrange*:

$$f = l_i^T(\sigma_v^2 + \psi_i)l_i - 2l_i^T\sigma_v^2 + \sigma_v^2 - 2\boldsymbol{\eta}^T(\mathbf{z}_i l_i - \boldsymbol{\gamma})$$
$$= l_i^T(\sigma_v^2 + \psi_i)l_i - 2l_i^T\sigma_v^2 + \sigma_v^2 - 2\boldsymbol{\eta}^T\mathbf{z}_i l_i + 2\boldsymbol{\eta}^T\boldsymbol{\gamma} \qquad \text{(viii)}$$

yang akan diturunkan terhadap l_i dan $\boldsymbol{\eta}$, dengan $\boldsymbol{\eta}$ adalah vektor pengali *Lagrange* yang berukuran $px1$, sehingga dari metode tersebut diperoleh l_i dan $\boldsymbol{\eta}$ yang meminimumkan fungsi f.

Berikut ini turunan parsial dari f terhadap l_i dan $\boldsymbol{\eta}$.

$$f = l_i^T(\sigma_v^2 + \psi_i)l_i - 2l_i^T\sigma_v^2 + \sigma_v^2 - 2\boldsymbol{\eta}^T\mathbf{z}_i l_i + 2\boldsymbol{\eta}^T\boldsymbol{\gamma}$$

a. Berikut akan dicari $\frac{\partial f}{\partial l_i}$ yang merupakan turunan parsial pertama dari f terhadap l_i. Untuk mendapat f yang minimum, maka $\frac{\partial f}{\partial l_i} = 0$.

$$\frac{\partial f}{\partial l_i} = \frac{\partial\left(l_i^T(\sigma_v^2 + \psi_i)l_i - 2l_i^T\sigma_v^2 + \sigma_v^2 - 2\boldsymbol{\eta}^T\mathbf{z}_i l_i + 2\boldsymbol{\eta}^T\boldsymbol{\gamma}\right)}{\partial l_i}$$
$$= \underbrace{\frac{\partial\left(l_i^T(\sigma_v^2 + \psi_i)l_i\right)}{\partial l_i}}_{1} - \underbrace{\frac{\partial\left(2l_i^T\sigma_v^2\right)}{\partial l_i}}_{2} + \underbrace{\frac{\partial(\sigma_v^2)}{\partial l_i}}_{3} - \underbrace{\frac{\partial(2\boldsymbol{\eta}^T\mathbf{z}_i l_i)}{\partial l_i}}_{4} + \underbrace{\frac{\partial(2\boldsymbol{\eta}^T\boldsymbol{\gamma})}{\partial l_i}}_{5}$$

Akan dicari nilai turunan masing-masing suku tersebut.

(1) $\frac{\partial\left(l_i^T(\sigma_v^2 + \psi_i)l_i\right)}{\partial l_i} = (\sigma_v^2 + \psi_i)\frac{\partial\left(l_i^2\right)}{\partial l_i} = 2(\sigma_v^2 + \psi_i)l_i$

(2) $\frac{\partial\left(2l_i^T\sigma_v^2\right)}{\partial l_i} = 2\sigma_v^2\frac{\partial\left(l_i^T\right)}{\partial l_i} = 2\sigma_v^2$

(3) $\frac{\partial(\sigma_v^2)}{\partial l_i} = 0$

(4) $\boldsymbol{\eta}^T\mathbf{z}_i = (\eta_1 \quad \cdots \quad \eta_p)\begin{pmatrix} z_{i1} \\ \vdots \\ z_{ip} \end{pmatrix} = \left(\eta_1 z_{i1} + \cdots + \eta_p z_{ip}\right)$

$$2\boldsymbol{\eta}^T\mathbf{z}_i l_i = 2\left(\eta_1 z_{i1} + \cdots + \eta_p z_{ip}\right)l_i$$

Lampiran 9 (lanjutan)

$$= \left(2\eta_1 z_{i1} + \cdots + 2\eta_p z_{ip}\right) l_i$$

$$\frac{\partial(2\boldsymbol{\eta}^T \mathbf{z}_i l_i)}{\partial l_i} = \frac{\partial\left(\left(2\eta_1 z_{i1} + \cdots + 2\eta_p z_{ip}\right) l_i\right)}{\partial l_i}$$

$$= \left(2\eta_1 z_{i1} + \cdots + 2\eta_p z_{ip}\right)$$

$$= 2\left(\eta_1 z_{i1} + \cdots + \eta_p z_{ip}\right)$$

$$= 2\boldsymbol{\eta}^T \mathbf{z}_i$$

(5) $\boldsymbol{\eta}^T \boldsymbol{\gamma} = (\eta_1 \quad \cdots \quad \eta_p) \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_p \end{pmatrix} = \left(\eta_1 \gamma_1 + \cdots + \eta_p \gamma_p\right)$

$$2\boldsymbol{\eta}^T \boldsymbol{\gamma} = 2\left(\eta_1 \gamma_1 + \cdots + \eta_p \gamma_p\right) = \left(2\eta_1 \gamma_1 + \cdots + 2\eta_p \gamma_p\right)$$

$$\frac{\partial(2\boldsymbol{\eta}^T \boldsymbol{\gamma})}{\partial l_i} = \frac{\partial\left(2\eta_1 \gamma_1 + \cdots + 2\eta_p \gamma_p\right)}{\partial l_i} = 0$$

sehingga didapat:

$$\frac{\partial f}{\partial l_i} = 2(\sigma_v^2 + \psi_i) l_i - 2\sigma_v^2 + 0 - 2\boldsymbol{\eta}^T \mathbf{z}_i + 0$$

$$= 2(\sigma_v^2 + \psi_i) l_i - 2\sigma_v^2 - 2\boldsymbol{\eta}^T \mathbf{z}_i$$

b. Berikut akan dicari $\frac{\partial f}{\partial \boldsymbol{\eta}}$ yang didefinisikan sebagai turunan parsial pertama dari f terhadap masing-masing elemen vektor $\boldsymbol{\eta}$. Agar didapat f yang minimum, maka $\frac{\partial f}{\partial \eta_1} = 0$, $\frac{\partial f}{\partial \eta_2} = 0$, … , $\frac{\partial f}{\partial \eta_p} = 0$, sehingga $\frac{\partial f}{\partial \boldsymbol{\eta}} = \mathbf{0}$.

$$\frac{\partial f}{\partial \boldsymbol{\eta}} = \frac{\partial\left({l_i}^T (\sigma_v^2 + \psi_i) l_i - 2{l_i}^T \sigma_v^2 + \sigma_v^2 - 2\boldsymbol{\eta}^T \mathbf{z}_i l_i + 2\boldsymbol{\eta}^T \boldsymbol{\gamma}\right)}{\partial \boldsymbol{\eta}}$$

$$= \underbrace{\frac{\partial\left({l_i}^T (\sigma_v^2 + \psi_i) l_i\right)}{\partial \boldsymbol{\eta}}}_{1} - \underbrace{\frac{\partial\left(2{l_i}^T \sigma_v^2\right)}{\partial \boldsymbol{\eta}}}_{2} + \underbrace{\frac{\partial(\sigma_v^2)}{\partial \boldsymbol{\eta}}}_{3} - \underbrace{\frac{\partial(2\boldsymbol{\eta}^T \mathbf{z}_i l_i)}{\partial \boldsymbol{\eta}}}_{4} + \underbrace{\frac{\partial(2\boldsymbol{\eta}^T \boldsymbol{\gamma})}{\partial \boldsymbol{\eta}}}_{5}$$

Akan dicari nilai turunan masing-masing suku tersebut.

(1) $\frac{\partial\left({l_i}^T (\sigma_v^2 + \psi_i) l_i\right)}{\partial \boldsymbol{\eta}} = (\sigma_v^2 + \psi_i) \frac{\partial\left({l_i}^2\right)}{\partial \boldsymbol{\eta}}$

Lampiran 9 (lanjutan)

$$\frac{\partial(l_i{}^2)}{\partial\boldsymbol{\eta}} = \begin{pmatrix} \frac{\partial(l_i{}^2)}{\partial\eta_1} \\ \vdots \\ \frac{\partial(l_i{}^2)}{\partial\eta_p} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

(2) $$\frac{\partial(2l_i\sigma_v^2)}{\partial\boldsymbol{\eta}} = 2\sigma_v^2 \frac{\partial(l_i{}^T)}{\partial\boldsymbol{\eta}} = 2\sigma_v^2 \begin{pmatrix} \frac{\partial(l_i{}^T)}{\partial\eta_1} \\ \vdots \\ \frac{\partial(l_i{}^T)}{\partial\eta_p} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

(3) $$\frac{\partial(\sigma_v^2)}{\partial\boldsymbol{\eta}} = \begin{pmatrix} \frac{\partial(\sigma_v^2)}{\partial\eta_1} \\ \vdots \\ \frac{\partial(\sigma_v^2)}{\partial\eta_p} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

(4) $$\boldsymbol{\eta}^{\boldsymbol{T}}\mathbf{z}_{\boldsymbol{i}} = (\eta_1 \quad \cdots \quad \eta_p)\begin{pmatrix} z_{i1} \\ \vdots \\ z_{ip} \end{pmatrix} = (\eta_1 z_{i1} + \cdots + \eta_p z_{ip})$$

$$2\boldsymbol{\eta}^{\boldsymbol{T}}\mathbf{z}_{\boldsymbol{i}} l_i = 2(\eta_1 z_{i1} + \cdots + \eta_p z_{ip})l_i$$

$$= (2\eta_1 z_{i1} + \cdots + 2\eta_p z_{ip})l_i$$

$$\frac{\partial(2\boldsymbol{\eta}^{\boldsymbol{T}}\mathbf{z}_{\boldsymbol{i}} l_i)}{\partial\boldsymbol{\eta}} = \frac{\partial(2\eta_1 z_{i1} l_i + \cdots + 2\eta_p z_{ip} l_i)}{\partial\boldsymbol{\eta}}$$

$$= \begin{pmatrix} \frac{\partial(2\eta_1 z_{i1} l_i + \cdots + 2\eta_p z_{ip} l_i)}{\partial\eta_1} \\ \vdots \\ \frac{\partial(2\eta_1 z_{i1} l_i + \cdots + 2\eta_p z_{ip} l_i)}{\partial\eta_p} \end{pmatrix}$$

$$= \begin{pmatrix} 2z_{i1} l_i \\ \vdots \\ 2z_{ip} l_i \end{pmatrix}$$

$$= 2\mathbf{z}_{\boldsymbol{i}} l_i$$

(5) $$\boldsymbol{\eta}^{\boldsymbol{T}}\boldsymbol{\gamma} = (\eta_1 \quad \cdots \quad \eta_p)\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_p \end{pmatrix} = (\eta_1\gamma_1 + \cdots + \eta_p\gamma_p)$$

$$2\boldsymbol{\eta}^{\boldsymbol{T}}\boldsymbol{\gamma} = 2(\eta_1\gamma_1 + \cdots + \eta_p\gamma_p) = (2\eta_1\gamma_1 + \cdots + 2\eta_p\gamma_p)$$

$$\frac{\partial(2\boldsymbol{\eta}^{\boldsymbol{T}}\boldsymbol{\gamma})}{\partial\boldsymbol{\eta}} = \frac{\partial(2\eta_1\gamma_1 + \cdots + 2\eta_p\gamma_p)}{\partial\boldsymbol{\eta}}$$

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$$= \begin{pmatrix} \dfrac{\partial(2\eta_1\gamma_1 + \cdots + 2\eta_p\gamma_p)}{\partial\eta_1} \\ \vdots \\ \dfrac{\partial(2\eta_1\gamma_1 + \cdots + 2\eta_p\gamma_p)}{\partial\eta_p} \end{pmatrix}$$

$$= \begin{pmatrix} 2\gamma_1 \\ \vdots \\ 2\gamma_p \end{pmatrix}$$

$$= 2\boldsymbol{\gamma}$$

sehingga didapat:

$$\frac{\partial f}{\partial \boldsymbol{\eta}} = \mathbf{0} - \mathbf{0} + \mathbf{0} - 2\mathbf{z}_i l_i + 2\boldsymbol{\gamma}$$

$$= -2\mathbf{z}_i l_i + 2\boldsymbol{\gamma}$$

Berdasarkan hasil yang diperoleh dari a) dan b), maka didapat:

$$\frac{\partial f}{\partial l_i} = 2(\sigma_v^2 + \psi_i)l_i - 2\sigma_v^2 - 2\boldsymbol{\eta}^T\mathbf{z}_i$$

$= 2(\sigma_v^2 + \psi_i)l_i - 2\sigma_v^2 - 2\mathbf{z}_i^T\boldsymbol{\eta}$; karena $\boldsymbol{\eta}^T\mathbf{z}_i = \mathbf{z}_i^T\boldsymbol{\eta} = (\eta_1 z_{i1} + \cdots + \eta_p z_{ip})$

$$\frac{\partial f}{\partial \boldsymbol{\eta}} = -2\mathbf{z}_i l_i + 2\boldsymbol{\gamma} \qquad \text{(ix)}$$

Berdasarkan metode *Lagrange*, nilai l_i dan $\boldsymbol{\eta}$ dapat diperoleh dari:

$$\frac{\partial f}{\partial l_i} = 2(\sigma_v^2 + \psi_i)l_i - 2\sigma_v^2 - 2\mathbf{z}_i^T\boldsymbol{\eta} = 0$$

$$\frac{\partial f}{\partial \boldsymbol{\eta}} = -2\mathbf{z}_i l_i + 2\boldsymbol{\gamma} = \mathbf{0} \qquad \text{(x)}$$

sehingga jika didapat persamaan:

$$(\sigma_v^2 + \psi_i)l_i - \mathbf{z}_i^T\boldsymbol{\eta} = \sigma_v^2$$

$$\mathbf{z}_i l_i - \boldsymbol{\gamma} = \mathbf{0} \qquad \text{(xi)}$$

Kemudian dibentuk menjadi persamaan:

$$\begin{bmatrix} (\sigma_v^2 + \psi_i) & \mathbf{z}_i^T \\ \mathbf{z}_i & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{l}_i \\ -\hat{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \sigma_v^2 \\ \boldsymbol{\gamma} \end{bmatrix} \qquad \text{(xii)}$$

Lampiran 9 (lanjutan)

Matriks $\mathbf{G}$, $\mathbf{R}$, dan $\mathbf{V}$ adalah matriks definit positif. Berdasarkan Lemma 2.1 yang telah disinggung pada subbab 2.3, sembarang matriks definit positif merupakan matriks *nonsingular*. Dengan demikian, matriks $\mathbf{G}$, $\mathbf{R}$, dan $\mathbf{V}$ adalah matriks *nonsingular*. Untuk mencari nilai $\widehat{l_i}$ dan $\widehat{\boldsymbol{\eta}}$ pada persamaan (xii), dapat digunakan teorema invers untuk matriks partisi, sehingga didapat:

$$\begin{bmatrix} \widehat{l_i} \\ -\widehat{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} (\sigma_v^2+\psi_i)^{-1}\left[\mathbf{I}-\mathbf{z_i^T}\left(\mathbf{z_i^{T^{-1}}}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_i(\sigma_v^2+\psi_i)^{-1}\right] & (\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\left(\mathbf{z_i^{T^{-1}}}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\right)^{-1} \\ \left(\boldsymbol{z_i}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\right)^{-1}\boldsymbol{z_i}(\sigma_v^2+\psi_i)^{-1} & -\left(\boldsymbol{z_i}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\right)^{-1} \end{bmatrix} \begin{bmatrix} \sigma_v^2 \\ \boldsymbol{\gamma} \end{bmatrix}$$

dengan

$$\widehat{l_i} = (\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\left(\mathbf{z_i^{T^{-1}}}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\right)^{-1}\boldsymbol{\gamma} + (\sigma_v^2+\psi_i)^{-1}\left[\mathbf{I}-\mathbf{z_i^T}\left(\mathbf{z_i^{T^{-1}}}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_i(\sigma_v^2+\psi_i)^{-1}\right]\sigma_v^2$$

$$-\widehat{\boldsymbol{\eta}} = (\boldsymbol{z_i}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T})^{-1}\boldsymbol{z_i}(\sigma_v^2+\psi_i)^{-1}\sigma_v^2 - (\boldsymbol{z_i}(\sigma_v^2+\psi_i)^{-1}\mathbf{z_i^T})^{-1}\boldsymbol{\gamma} \qquad \text{(xiii)}$$

Karena sembarang penaksir $T(\hat{\theta}_i) = {l_i}^T\hat{\theta}_i$ memenuhi sifat *best*, linear, dan *unbiased*, maka:

$$T(\hat{\theta}_i) = {l_i}^T\hat{\theta}_i$$

$$= \begin{Bmatrix} \left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z_i^T}\left(\mathbf{z_i^{T^{-1}}}\left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\boldsymbol{\gamma} + \left(\sigma_v^2+\psi_i\right)^{-1} \\ \left[\mathbf{I}-\mathbf{z_i^T}\left(\mathbf{z_i^{T^{-1}}}\left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\boldsymbol{z_i}\left(\sigma_v^2+\psi_i\right)^{-1}\right]\sigma_v^2 \end{Bmatrix}^T \hat{\theta}_i$$

$$= \boldsymbol{\gamma}^{\boldsymbol{T}}\underbrace{\left(\boldsymbol{z_i}\left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\boldsymbol{z_i}\left(\sigma_v^2+\psi_i\right)^{-1}\hat{\theta}_i}_{\widehat{\boldsymbol{\beta}}} + \sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}\hat{\theta}_i - \sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z_i^T}\underbrace{\left(\boldsymbol{z_i}\left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\boldsymbol{z_i}\left(\sigma_v^2+\psi_i\right)^{-1}\hat{\theta}_i}_{\widehat{\boldsymbol{\beta}}}$$

Lampiran 9 (lanjutan)

dengan $(\mathbf{z}_i(\sigma_v^2+\psi_i)^{-1}\mathbf{z}_i^{\mathbf{T}})^{-1}\mathbf{z}_i(\sigma_v^2+\psi_i)^{-1}\hat{\theta}_i = \widehat{\boldsymbol{\beta}}$, persamaan sebelumnya dapat ditulis:

$$= \boldsymbol{\gamma}^T\widehat{\boldsymbol{\beta}} + \sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}\hat{\theta}_i - \sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}$$

$$= \boldsymbol{\gamma}^T\widehat{\boldsymbol{\beta}} + \sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}})$$

Berdasarkan hasil tersebut, maka taksiran BLUP untuk kombinasi linear $\boldsymbol{\gamma}^{\mathbf{T}}\boldsymbol{\beta} + v_i$, adalah:

$$\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2) = \boldsymbol{\gamma}^T\widehat{\boldsymbol{\beta}} + \sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}})$$

$$= \boldsymbol{\gamma}^T\widehat{\boldsymbol{\beta}} + \hat{v}_i \qquad \text{(xiv)}$$

Oleh karena taksiran BLUP masih bergantung pada variansi pengaruh acak (σ_v^2) yang tidak diketahui, maka taksiran parameter BLUP $\widehat{\boldsymbol{\beta}}(\sigma_v^2)$ dan $\hat{v}_i(\sigma_v^2)$ yang diperoleh adalah:

$$\widehat{\boldsymbol{\beta}}(\sigma_v^2) = \left(\mathbf{z}_i\left(\sigma_v^2+\psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\sigma_v^2+\psi_i\right)^{-1}\hat{\theta}_i$$

$$\hat{v}_i(\sigma_v^2) = \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right).$$

Lampiran 10. Bukti $\left(\mathbf{z_i^T\beta}+\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i-\mathbf{z_i^T\beta}\right)-\left(\mathbf{z_i^T\beta}+v_i\right)\right)$ dan $\left(\left(\mathbf{z_i^T}-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)$ Independen

Berikut akan ditunjukkan bahwa $\left(\mathbf{z_i^T\beta}+\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i-\mathbf{z_i^T\beta}\right)-\left(\mathbf{z_i^T\beta}+v_i\right)\right)$ dan $\left(\left(\mathbf{z_i^T}-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)$ independen. $\left(\mathbf{z_i^T\beta}+\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i-\mathbf{z_i^T\beta}\right)-\left(\mathbf{z_i^T\beta}+v_i\right)\right)$ dan $\left(\left(\mathbf{z_i^T}-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)$ dikatakan independen jika:

$cov\left[\left(\mathbf{z_i^T\beta}+\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i-\mathbf{z_i^T\beta}\right)-\left(\mathbf{z_i^T\beta}+v_i\right)\right),\left(\left(\mathbf{z_i^T}-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)\right]=0.$

Bukti:

$$cov\left[\left(\mathbf{z_i^T\beta}+\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i-\mathbf{z_i^T\beta}\right)-\left(\mathbf{z_i^T\beta}+v_i\right)\right),\left(\left(\mathbf{z_i^T}-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\mathbf{z_i^T}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)\right]$$

$= cov\left[\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)-v_i\right),\left(\left(1-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)\mathbf{z_i^T}\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)\right]$; karena $\hat{\theta}_i-\mathbf{z_i^T\beta}=v_i+e_i$

$$= cov\left[\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)-v_i\right),\left(\frac{\psi_i\mathbf{z_i^T}}{(\sigma_v^2+\psi_i)}\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)\right]$$

Lampiran 10 (lanjutan)

$$= E\left[\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)-v_i\right)-\underbrace{E\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)-v_i\right)}_{(i)}\right]\left[\left(\frac{\psi_i\mathbf{z_i^T}}{(\sigma_v^2+\psi_i)}\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)-\underbrace{E\left(\frac{\psi_i\mathbf{z_i^T}}{(\sigma_v^2+\psi_i)}\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)}_{(ii)}\right]^T$$

Akan dicari nilai (i) dan (ii) terlebih dahulu:

$$(i)\ E\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)-v_i\right)=E\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)\right)-E(v_i)$$

$$=\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}E(v_i+e_i)-E(v_i)$$

$$=\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\{E(v_i)+E(e_i)\}-E(v_i)$$

$$=\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(0+0)-0$$

$$=0.$$

karena $\widehat{\boldsymbol{\beta}}(\sigma_v^2)=\left(\left(\frac{\mathbf{z_i z_i^T}}{(\sigma_v^2+\psi_i)}\right)^{-1}\frac{\mathbf{z_i}\hat{\theta}_i}{(\sigma_v^2+\psi_i)}\right)$ dan $\boldsymbol{\beta}=\frac{\hat{\theta}_i-v_i-e_i}{\mathbf{z_i^T}}$ maka:

Lampiran 10 (lanjutan)

$$\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right) = \left(\left(\frac{\mathbf{z_i z_i^T}}{(\sigma_v^2 + \psi_i)}\right)^{-1} \frac{\mathbf{z_i}\hat{\theta}_i}{(\sigma_v^2 + \psi_i)} - \frac{\hat{\theta}_i - v_i - e_i}{\mathbf{z_i^T}}\right)$$

$$= \frac{\dfrac{\mathbf{z_i}\hat{\theta}_i}{(\sigma_v^2 + \psi_i)}}{\dfrac{\mathbf{z_i z_i^T}}{(\sigma_v^2 + \psi_i)}} - \left(\frac{\mathbf{z_i}}{\mathbf{z_i}} . \frac{\hat{\theta}_i - v_i - e_i}{\mathbf{z_i^T}}\right)$$

$$= \frac{\mathbf{z_i}\hat{\theta}_i}{\mathbf{z_i z_i^T}} - \frac{\mathbf{z_i}\hat{\theta}_i - \mathbf{z_i}v_i - \mathbf{z_i}e_i}{\mathbf{z_i z_i^T}}$$

$$= \frac{\mathbf{z_i}\hat{\theta}_i - \mathbf{z_i}\hat{\theta}_i + \mathbf{z_i}v_i + \mathbf{z_i}e_i}{\mathbf{z_i z_i^T}}$$

$$= \frac{\mathbf{z_i}(v_i + e_i)}{\mathbf{z_i z_i^T}}$$

$$(ii) E\left(\frac{\psi_i \mathbf{z_i^T}}{(\sigma_v^2 + \psi_i)}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right) = E\left(\frac{\psi_i \mathbf{z_i^T}}{(\sigma_v^2 + \psi_i)} \frac{\mathbf{z_i}(v_i + e_i)}{\mathbf{z_i z_i^T}}\right)$$

$$= E\left(\frac{\psi_i}{(\sigma_v^2 + \psi_i)} \frac{\mathbf{z_i^T z_i}(v_i + e_i)}{\mathbf{z_i z_i^T}}\right)$$

$$= \frac{\psi_i}{(\sigma_v^2 + \psi_i)} \underbrace{\frac{\mathbf{z_i^T z_i}}{\mathbf{z_i z_i^T}}}_{=1} E(v_i + e_i)$$

$$= \frac{\psi_i}{(\sigma_v^2 + \psi_i)} \{E(v_i) + E(e_i)\}$$

$$= \frac{\psi_i}{(\sigma_v^2 + \psi_i)} (0 + 0)$$

$$= 0.$$

Dengan nilai (i) dan (ii) yang diperoleh, maka:

$$= E\left[\left(\frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}(v_i + e_i) - v_i\right) - 0\right]\left[\left(\frac{\psi_i \mathbf{z_i^T}}{(\sigma_v^2 + \psi_i)}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right) - 0\right]^T$$

$$= E\left[\frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}(v_i + e_i) - v_i\right]\left[\frac{\psi_i \mathbf{z_i^T}}{(\sigma_v^2 + \psi_i)} \frac{\mathbf{z_i}(v_i + e_i)}{\mathbf{z_i z_i^T}}\right]^T$$

$$= E\left[\frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)} v_i + \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)} e_i - v_i\right]\left[\frac{\psi_i}{(\sigma_v^2 + \psi_i)} \underbrace{\frac{\mathbf{z_i^T z_i}}{\mathbf{z_i z_i^T}}}_{=1} (v_i + e_i)\right]^T$$

$$= E\left[\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)} - 1\right) v_i + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)} e_i\right]\left[\frac{\psi_i}{(\sigma_v^2+\psi_i)} v_i{}^T + \frac{\psi_i}{(\sigma_v^2+\psi_i)} e_i{}^T\right]$$

$$= E\left[\frac{-\psi_i}{(\sigma_v^2+\psi_i)} v_i + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)} e_i\right]\left[\frac{\psi_i}{(\sigma_v^2+\psi_i)} v_i{}^T + \frac{\psi_i}{(\sigma_v^2+\psi_i)} e_i{}^T\right]$$

$$= E\left[\frac{-\psi_i}{(\sigma_v^2+\psi_i)} v_i \frac{\psi_i}{(\sigma_v^2+\psi_i)} v_i{}^T\right] + E\left[\frac{-\psi_i}{(\sigma_v^2+\psi_i)} v_i \frac{\psi_i}{(\sigma_v^2+\psi_i)} e_i{}^T\right]$$

$$+ E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)} e_i \frac{\psi_i}{(\sigma_v^2+\psi_i)} v_i{}^T\right] + E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)} e_i \frac{\psi_i}{(\sigma_v^2+\psi_i)} e_i{}^T\right]$$

$$= \frac{-\psi_i{}^2}{(\sigma_v^2+\psi_i)^2} E[v_i v_i{}^T] + \frac{-\psi_i{}^2}{(\sigma_v^2+\psi_i)^2} E[v_i e_i{}^T] + \frac{\sigma_v^2\psi_i}{(\sigma_v^2+\psi_i)^2} E[e_i v_i{}^T]$$

$$+ \frac{\sigma_v^2\psi_i}{(\sigma_v^2+\psi_i)^2} E[e_i e_i{}^T]$$

$$= \frac{-\psi_i{}^2}{(\sigma_v^2+\psi_i)^2}(\sigma_v^2) + \frac{-\psi_i{}^2}{(\sigma_v^2+\psi_i)^2}(0) + \frac{\sigma_v^2\psi_i}{(\sigma_v^2+\psi_i)^2}(0) + \frac{\sigma_v^2\psi_i}{(\sigma_v^2+\psi_i)^2}(\psi_i)$$

$$= \frac{-\psi_i{}^2\sigma_v^2}{(\sigma_v^2+\psi_i)^2} + \frac{\sigma_v^2\psi_i{}^2}{(\sigma_v^2+\psi_i)^2}$$

$$= 0.$$

Jadi terbukti bahwa $\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i - \mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}\right) - \left(\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta} + v_i\right)\right)$ dan $\left(\left(\mathbf{z}_\mathbf{i}^\mathbf{T} - \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right)$ independen.

Lampiran 11. Bukti $g_1(\sigma_v^2) = \frac{\sigma_v^2 \psi_i}{(\sigma_v^2+\psi_i)}$ dan

$g_2(\sigma_v^2) = \left(1 - \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)^2 \mathbf{z_i^T} \left(\frac{(\sigma_v^2+\psi_i)}{\mathbf{z_i z_i^T}}\right) \mathbf{z_i}$

Akan ditunjukkan bahwa:

1. $g_1(\sigma_v^2) = \frac{\sigma_v^2 \psi_i}{(\sigma_v^2+\psi_i)}$
2. $g_2(\sigma_v^2) = \left(1 - \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)^2 \mathbf{z_i^T} \left(\frac{(\sigma_v^2+\psi_i)}{\mathbf{z_i z_i^T}}\right) \mathbf{z_i}$

Bukti:

$$g_1(\sigma_v^2) = E\left[\mathbf{z_i^T\beta} + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T\beta}\right) - \left(\mathbf{z_i^T\beta} + v_i\right)\right]^2$$

$$= E\left[\mathbf{z_i^T\beta} + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i) - \mathbf{z_i^T\beta} - v_i\right]^2 \text{ ; karena } \hat{\theta}_i - \mathbf{z_i^T\beta} = v_i + e_i$$

$$= E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i) - v_i\right]^2$$

$$= E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i) - v_i\right]\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i) - v_i\right]^T$$

$$= E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i) - v_i\right]\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)^T - {v_i}^T\right]$$

$$= E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i) - v_i\right]\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}({v_i}^T+{e_i}^T) - {v_i}^T\right]$$

$$= E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i)\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}({v_i}^T+{e_i}^T)\right] - E\left[\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}(v_i+e_i){v_i}^T\right]$$
$$-E\left[v_i\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}({v_i}^T+{e_i}^T)\right] + E[v_i{v_i}^T]$$

$$= \left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)^2 E[v_i{v_i}^T + v_i{e_i}^T + e_i{v_i}^T + e_i{e_i}^T] - \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}E[v_i{v_i}^T + e_i{v_i}^T]$$
$$-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}E[v_i{v_i}^T + v_i{e_i}^T] + E[v_i{v_i}^T]$$

Lampiran 11 (lanjutan)

$$= \left(\frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \{E[v_i v_i^T] + E[v_i e_i^T] + E[e_i v_i^T] + E[e_i e_i^T]\}$$

$$- \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\{E[v_i v_i^T] + E[e_i v_i^T]\} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\{E[v_i v_i^T] + E[v_i e_i^T]\} + E[v_i v_i^T]$$

$$= \left(\frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \{\sigma_v^2 + 0 + 0 + \psi_i\} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\{\sigma_v^2 + 0\} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\{\sigma_v^2 + 0\} + \sigma_v^2$$

$$= \left(\frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 (\sigma_v^2 + \psi_i) - 2\frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}(\sigma_v^2) + \sigma_v^2$$

$$= \frac{(\sigma_v^2)^2}{(\sigma_v^2 + \psi_i)} - 2\frac{(\sigma_v^2)^2}{(\sigma_v^2 + \psi_i)} + \sigma_v^2$$

$$= \sigma_v^2 - \frac{(\sigma_v^2)^2}{(\sigma_v^2 + \psi_i)}$$

$$= \frac{\sigma_v^2(\sigma_v^2 + \psi_i) - (\sigma_v^2)^2}{(\sigma_v^2 + \psi_i)}$$

$$= \frac{(\sigma_v^2)^2 + \sigma_v^2\psi_i - (\sigma_v^2)^2}{(\sigma_v^2 + \psi_i)}$$

$$= \frac{\sigma_v^2\psi_i}{(\sigma_v^2 + \psi_i)} \quad \text{(terbukti).}$$

$$g_2(\sigma_v^2) = E\left[\left(\mathbf{z_i^T} - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\mathbf{z_i^T}\right)\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]^2$$

$$= E\left[\left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]^2$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 E\left[\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]^2$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 E\left[\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]\left[\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right]^T$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 E\left[\left(\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right)\left(\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\right)^T\right]$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 E\left[\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)^T\mathbf{z_i}\right]$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \mathbf{z_i^T} E\left[\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\left(\hat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)^T\right]\mathbf{z_i}$$

Lampiran 11 (lanjutan)

Karena $\widehat{\boldsymbol{\beta}}(\sigma_v^2) = \left(\left(\frac{\mathbf{z_i}\mathbf{z_i^T}}{(\sigma_v^2+\psi_i)}\right)^{-1}\frac{\mathbf{z_i}\hat{\theta}_i}{(\sigma_v^2+\psi_i)}\right)$ dan $\boldsymbol{\beta} = \frac{\hat{\theta}_i - v_i - e_i}{\mathbf{z_i^T}}$ maka

$$\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right) = \left(\left(\frac{\mathbf{z_i}\mathbf{z_i^T}}{(\sigma_v^2+\psi_i)}\right)^{-1}\frac{\mathbf{z_i}\hat{\theta}_i}{(\sigma_v^2+\psi_i)} - \frac{\hat{\theta}_i - v_i - e_i}{\mathbf{z_i^T}}\right)$$

$$= \frac{\frac{\mathbf{z_i}\hat{\theta}_i}{(\sigma_v^2+\psi_i)}}{\frac{\mathbf{z_i}\mathbf{z_i^T}}{(\sigma_v^2+\psi_i)}} - \left(\frac{\mathbf{z_i}}{\mathbf{z_i}}.\frac{\hat{\theta}_i - v_i - e_i}{\mathbf{z_i^T}}\right)$$

$$= \frac{\mathbf{z_i}\hat{\theta}_i}{\mathbf{z_i}\mathbf{z_i^T}} - \frac{\mathbf{z_i}\hat{\theta}_i - \mathbf{z_i}v_i - \mathbf{z_i}e_i}{\mathbf{z_i}\mathbf{z_i^T}}$$

$$= \frac{\mathbf{z_i}\hat{\theta}_i - \mathbf{z_i}\hat{\theta}_i + \mathbf{z_i}v_i + \mathbf{z_i}e_i}{\mathbf{z_i}\mathbf{z_i^T}}$$

$$= \frac{\mathbf{z_i}(v_i + e_i)}{\mathbf{z_i}\mathbf{z_i^T}}$$

$$\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)^T = \left(\frac{\mathbf{z_i}(v_i + e_i)}{\mathbf{z_i}\mathbf{z_i^T}}\right)^T = \frac{(v_i^T + e_i^T)\mathbf{z_i^T}}{\mathbf{z_i^T}\mathbf{z_i}}$$

$$E\left[\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \boldsymbol{\beta}\right)^T\right] = E\left[\left(\frac{\mathbf{z_i}(v_i + e_i)}{\mathbf{z_i}\mathbf{z_i^T}}\right)\left(\frac{(v_i^T + e_i^T)\mathbf{z_i^T}}{\mathbf{z_i^T}\mathbf{z_i}}\right)\right]$$

$$= E\left[\frac{\mathbf{z_i}(v_i + e_i)(v_i^T + e_i^T)\mathbf{z_i^T}}{\mathbf{z_i}\mathbf{z_i^T}\mathbf{z_i^T}\mathbf{z_i}}\right]$$

$$= \frac{\mathbf{z_i}E\left[v_i v_i^T + v_i e_i^T + e_i v_i^T + e_i e_i^T\right]\mathbf{z_i^T}}{\mathbf{z_i}\mathbf{z_i^T}\mathbf{z_i^T}\mathbf{z_i}}$$

$$= \left(\frac{\mathbf{z_i}(\sigma_v^2 + \psi_i)\mathbf{z_i^T}}{\mathbf{z_i}\mathbf{z_i^T}\mathbf{z_i^T}\mathbf{z_i}}\right)$$

sehingga diperoleh:

$$g_2(\sigma_v^2) = \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \mathbf{z_i^T}\left(\frac{\mathbf{z_i}(\sigma_v^2 + \psi_i)\mathbf{z_i^T}}{\mathbf{z_i}\mathbf{z_i^T}\mathbf{z_i^T}\mathbf{z_i}}\right)\mathbf{z_i}$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 (\sigma_v^2 + \psi_i)\frac{\mathbf{z_i^T}\mathbf{z_i}}{\mathbf{z_i}\mathbf{z_i^T}}\underbrace{\left(\frac{\mathbf{z_i^T}\mathbf{z_i}}{\mathbf{z_i^T}\mathbf{z_i}}\right)}_{=1}$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \frac{\mathbf{z_i^T}(\sigma_v^2 + \psi_i)\mathbf{z_i}}{\mathbf{z_i}\mathbf{z_i^T}}$$

$$= \left(1 - \frac{\sigma_v^2}{(\sigma_v^2 + \psi_i)}\right)^2 \mathbf{z_i^T}\left(\frac{(\sigma_v^2 + \psi_i)}{\mathbf{z_i}\mathbf{z_i^T}}\right)\mathbf{z_i} \quad \text{(terbukti).}$$

Lampiran 12. Bukti $\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]$ dan $\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]$ Independen

Akan ditunjukkan bahwa $\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]$ dan $\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]$ independen, yaitu:

$$cov\left\{\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right], \left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]\right\} = 0.$$

dengan diberikan $\widehat{\sigma_v^2}$ *translation-invariant* dan diasumsikan bahwa v_i dan e_i berdistribusi normal.

Bukti:

Diketahui bahwa jika distribusi bersama dari $\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]$ dan $\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$ adalah multivariat normal, maka $\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]$ independen terhadap $\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$ dan $\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]$ independen terhadap setiap fungsi dari $\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$.

Kemudian, dengan diberikan $\widehat{\sigma_v^2}$ *translation-invariant* dan diasumsikan bahwa v_i dan e_i berdistribusi normal, maka akan ditunjukkan terlebih dahulu bahwa $\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]$ merupakan fungsi dari $\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$.

$$\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right] = \left[\left(\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\widehat{\sigma_v^2}) + \hat{\boldsymbol{v}}(\widehat{\sigma_v^2})\right) - \left(\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2) + \hat{\boldsymbol{v}}(\sigma_v^2)\right)\right]$$

$$= \left[\mathbf{z_i^T}\underbrace{\left(\widehat{\boldsymbol{\beta}}(\widehat{\sigma_v^2}) - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)}_{1} + \underbrace{\left(\hat{\boldsymbol{v}}(\widehat{\sigma_v^2}) - \hat{\boldsymbol{v}}(\sigma_v^2)\right)}_{2}\right]$$

dimana bentuk 1 dan 2, sebagai berikut:

1. $\left(\widehat{\boldsymbol{\beta}}(\widehat{\sigma_v^2}) - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)$

$$= \left(\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i - \left(\mathbf{z}_{\boldsymbol{i}}(\sigma_v^2 + \psi_i)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}(\sigma_v^2 + \psi_i)^{-1}\hat{\theta}_i$$

$$= \left(\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i - \left(\mathbf{z}_{\boldsymbol{i}}(\sigma_v^2 + \psi_i)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}(\sigma_v^2 + \psi_i)^{-1}\hat{\theta}_i$$
$$+\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \widehat{\boldsymbol{\beta}}(\sigma_v^2)$$

Lampiran 12 (lanjutan)

$$= \left[\left(\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right]$$

$$- \left[\left(\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\hat{\theta}_i - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right]$$

$$= \begin{bmatrix} \left(\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i \\ -\left(\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2) \end{bmatrix}$$

$$- \begin{bmatrix} \left(\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\hat{\theta}_i \\ -\left(\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2) \end{bmatrix}$$

$$= \left[\left(\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\right]$$

$$- \left[\left(\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\right]$$

$$= \begin{bmatrix} \left(\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1} \\ -\left(\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i(\sigma_v^2 + \psi_i)^{-1} \end{bmatrix}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)$$

2. $\left(\hat{\boldsymbol{v}}\left(\widehat{\sigma_v^2}\right) - \hat{\boldsymbol{v}}(\sigma_v^2)\right)$

$$= \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}\left(\widehat{\sigma_v^2}\right)\right) - \sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)$$

$$= \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i - \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}\left(\widehat{\sigma_v^2}\right)$$

$$-\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)$$

$$= \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i$$

$$-\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\left(\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i$$

$$-\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)$$

$$= \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i$$

$$-\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\left(\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\right)^{-1}\mathbf{z}_i\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\hat{\theta}_i$$

$$-\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) + \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$$

$$-\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2} + \psi_i\right)^{-1}\mathbf{z}_i^{\mathbf{T}}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$$

Lampiran 11 (lanjutan)

$$
\begin{aligned}
&= \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\hat{\theta}_i - \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2) \\
&\quad -\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\left(\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\hat{\theta}_i \\
&\quad +\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2) - \sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) \\
&= \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\hat{\theta}_i - \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2) \\
&\quad -\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\left(\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\hat{\theta}_i \\
&\quad +\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\left(\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2) \\
&\quad -\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) \\
&= \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\left(\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) \\
&\quad -\widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\left(\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\left(\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) \\
&\quad -\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) \\
&= \left[\begin{array}{c} \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1} - \widehat{\sigma_v^2}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\left(\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1}\mathbf{z_i^T}\right)^{-1}\mathbf{z}_{\boldsymbol{i}}\left(\widehat{\sigma_v^2}+\psi_i\right)^{-1} \\ -\sigma_v^2(\sigma_v^2+\psi_i)^{-1} \end{array}\right] \\
&\quad \left(\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)
\end{aligned}
$$

Berdasarkan 1 dan 2, terlihat bahwa $\left(\widehat{\boldsymbol{\beta}}\left(\widehat{\sigma_v^2}\right) - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)$ dan $\left(\widehat{\boldsymbol{v}}\left(\widehat{\sigma_v^2}\right) - \widehat{\boldsymbol{v}}(\sigma_v^2)\right)$ merupakan fungsi yang bergantung pada $\hat{\theta}_i$ yaitu pada $\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$, sehingga $\left[\left(T\left(\hat{\theta}_i\right)\right)\left(\widehat{\sigma_v^2}\right) - \left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2)\right]$ merupakan fungsi yang *translation-invariant*.

Ambil $\hat{\theta}_i - \mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)$ adalah *maximal-invariant* dan setiap fungsi yang *translation-invariant* dari $\hat{\theta}_i$ dapat dinyatakan sebagai fungsi *maximal-invariant*. Berikut akan ditunjukkan bahwa $\left[\left(T\left(\hat{\theta}_i\right)\right)\left(\widehat{\sigma_v^2}\right) - \left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2)\right]$ dan $\left[\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2) - \theta_i\right]$ independen.

Lampiran 12 (lanjutan)

$$cov\left\{\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)-\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right],\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)-\theta_i\right]\right\}$$

$$= cov\left\{\left[\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2)\right],\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)-\theta_i\right]\right\}$$

$$= E\left\{\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2)\right)-\underbrace{E\left[\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2)\right]}_{(i)}\right\}$$

$$\left\{\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)-\theta_i\right)-\underbrace{E\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)-\theta_i\right]}_{(ii)}\right\}^T$$

Berikut perhitungan untuk (i) dan (ii):

$(i).\ E\left[\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2)\right]=E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}+v_i+e_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2)\right]$

$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}(\sigma_v^2)\right)+v_i+e_i\right]$

$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(-\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)\right)+v_i+e_i\right]$; karena $\hat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}=\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)$

$= E\left[-\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)+v_i+e_i\right]$

$= -\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}E[v_i+e_i]+E[v_i]+E[e_i]$

$= -\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}.0+0+0$

$= 0.$

$(ii).\ E\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)-\theta_i\right]$

$= E\left[\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2)+\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2)\right)\right)-\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}+v_i\right)\right]$

$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)+\sigma_v^2(\sigma_v^2+\psi_i)^{-1}(\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}+v_i+e_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\hat{\boldsymbol{\beta}}(\sigma_v^2))-v_i\right]$

$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)+\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}(\sigma_v^2)\right)+v_i+e_i\right)-v_i\right]$

$$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)\right)+\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\begin{pmatrix}\mathbf{z}_\mathbf{i}^\mathbf{T}\left(-\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)\right)\\ +v_i+e_i\end{pmatrix}-v_i\right]$$

; karena $\hat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}=\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)$

$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)-\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)+v_i+e_i\right)-v_i\right]$

$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)\right]$

$-E\left[\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i+e_i)+v_i+e_i\right)\right]-E[v_i]$

Lampiran 12 (lanjutan)

$$= \mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}E[v_i+e_i]-\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(E[v_i+e_i]+E[v_i]+E[e_i])$$
$$-E[v_i]$$
$$= \mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}.0-\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(0+0+0)-0$$
$$= 0.$$

Dari persamaan (i) dan (ii), maka didapat:

$$cov\left\{\left[\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right],\left[\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2)-\theta_i\right]\right\}$$
$$= E\left[\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right]\left[\left(T\left(\hat{\theta}_i\right)\right)(\sigma_v^2)-\theta_i\right]^T$$
$$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}+v_i+e_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right]$$
$$\left[\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)+\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\right)-\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}+v_i\right)\right]^T$$
$$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)+v_i+e_i\right]$$
$$\left[\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)+\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\right)-v_i\right]^T$$
$$= E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)+(v_i+e_i)\right]$$
$$\left[\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\mathbf{z}_{\boldsymbol{i}}+\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)-v_i\right)^T\right]$$
$$= E\begin{bmatrix} \mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\mathbf{z}_{\boldsymbol{i}} \\ +\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)-v_i\right)^T+(v_i+e_i)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\mathbf{z}_{\boldsymbol{i}} \\ +(v_i+e_i)\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)-v_i\right)^T \end{bmatrix}$$
$$= \underbrace{E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\mathbf{z}_{\boldsymbol{i}}\right]}_{I}$$
$$+\underbrace{E\left[\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)-v_i\right)^T\right]}_{II}$$
$$+\underbrace{E\left[(v_i+e_i)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\mathbf{z}_{\boldsymbol{i}}\right]}_{III}$$
$$+\underbrace{E\left[(v_i+e_i)\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)-v_i\right)^T\right]}_{IV}$$

Berikut akan diberikan penjabaran dari masing-masing nilai ekspektasi tersebut:

Lampiran 12 (lanjutan)

$$I.\ E\left[\mathbf{z_i^T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\mathbf{z}_i\right]$$

$$=\mathbf{z_i^T}E\left[\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\right]\mathbf{z}_i$$

$$=\mathbf{z_i^T}E\left[\left(-\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\right)\left(\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\right)^T\right]\mathbf{z}_i$$

$$=\mathbf{z_i^T}E\left[-\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)(v_i+e_i)^T\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\mathbf{z_i^T}\right]\mathbf{z}_i$$

$$=-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}E[(v_i+e_i)(v_i+e_i)^T]\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\mathbf{z_i^T}\mathbf{z}_i$$

$$=-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\left(E[v_iv_i{}^T]+E[v_ie_i{}^T]+E[e_iv_i{}^T]+E[e_ie_i{}^T]\right)\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\mathbf{z_i^T}\mathbf{z}_i$$

$$=-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(\sigma_v^2+0+0+\psi_i)\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\mathbf{z_i^T}\mathbf{z}_i$$

$$=-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(\sigma_v^2+\psi_i)\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\mathbf{z_i^T}\mathbf{z}_i$$

$$=-(\sigma_v^2+\psi_i)\frac{\mathbf{z_i^T}}{\mathbf{z}_i\mathbf{z_i^T}}\underbrace{\frac{\mathbf{z}_i\mathbf{z_i^T}}{\mathbf{z}_i\mathbf{z_i^T}}}_{=1}\mathbf{z}_i$$

$$=-(\sigma_v^2+\psi_i)\frac{\mathbf{z_i^T}\mathbf{z}_i}{\mathbf{z}_i\mathbf{z_i^T}}$$

$$=-(\sigma_v^2+\psi_i)\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\mathbf{z_i^T}\mathbf{z}_i$$

$$II.\ E\left[\mathbf{z_i^T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\hat{\theta}_i-\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)-v_i\right)^T\right]$$

$$=E\left[\mathbf{z_i^T}\left(-\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\right)\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\mathbf{z_i^T}\boldsymbol{\beta}+v_i+e_i-\mathbf{z_i^T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)-v_i\right)^T\right]$$

; karena $\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}=\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)$

$$=E\left[-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\left(\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\mathbf{z_i^T}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)+v_i+e_i\right)-v_i\right)^T\right]$$

$$=E\left[-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\begin{pmatrix}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\\ \left(\mathbf{z_i^T}\left(-\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\right)+v_i+e_i\right)-v_i\end{pmatrix}^T\right]$$

$$=E\left[-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\begin{pmatrix}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\\ \left(-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)+v_i+e_i\right)^T-v_i{}^T\end{pmatrix}\right]$$

$$=E\left[-\mathbf{z_i^T}\mathbf{z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}(v_i+e_i)\begin{pmatrix}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\\ \left(-(v_i+e_i)^T\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\mathbf{z_i^T}\mathbf{z}_i+v_i{}^T+e_i{}^T\right)-v_i{}^T\end{pmatrix}\right]$$

Lampiran 12 (lanjutan)

$$= E\left[\begin{array}{c}\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}(v_i+e_i)(v_i+e_i)^T\left(\mathbf{z_i z_i^T}\right)^{-1}\mathbf{z_i^T z_i} \\ -\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}(v_i+e_i)({v_i}^T+{e_i}^T)+\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}(v_i+e_i){v_i}^T\end{array}\right]$$

$$= E\left[\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}(v_i+e_i)(v_i+e_i)^T\left(\mathbf{z_i z_i^T}\right)^{-1}\mathbf{z_i^T z_i}\right]$$

$$-E\left[\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}(v_i+e_i)({v_i}^T+{e_i}^T)\right]$$

$$+E\left[\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}(v_i+e_i){v_i}^T\right]$$

$$= \mathbf{z_i^T z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}E[(v_i+e_i)(v_i+e_i)^T]\left(\mathbf{z_i z_i^T}\right)^{-1}\mathbf{z_i^T z}_i$$

$$-\mathbf{z_i^T z}_i\left(\mathbf{z}_i\mathbf{z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}E[(v_i+e_i)({v_i}^T+{e_i}^T)]$$

$$+\mathbf{z_i^T z}_i\left(\mathbf{z_i z_i^T}\right)^{-1}E[(v_i+e_i){v_i}^T]$$

$$= \mathbf{z_i^T z}_i(\mathbf{z_i z_i^T})^{-1}\sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}(E[v_i{v_i}^T]+E[v_i{e_i}^T]+E[e_i{v_i}^T]+E[e_i{e_i}^T])$$

$$\left(\mathbf{z_i z_i^T}\right)^{-1}\mathbf{z_i^T z_i}-\mathbf{z_i^T z}_i\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}$$

$$(E[v_i{v_i}^T]+E[v_i{e_i}^T]+E[e_i{v_i}^T]+E[e_i{e_i}^T])+\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}E[(v_i+e_i){v_i}^T]$$

$$= \mathbf{z_i^T z_i}(\mathbf{z_i z_i^T})^{-1}\sigma_v^2\left(\sigma_v^2+\psi_i\right)^{-1}(\sigma_v^2+0+0+\psi_i)(\mathbf{z_i z_i^T})^{-1}\mathbf{z_i^T z}_i$$

$$-\mathbf{z_i^T z}_i\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2(\sigma_v^2+\psi_i)^{-1}\left(\sigma_v^2+0+0+\psi_i\right)+\mathbf{z_i^T z}_i\left(\mathbf{z_i z_i^T}\right)^{-1}E[(v_i+e_i){v_i}^T]$$

$$= \mathbf{z_i^T z}_i\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2\underbrace{(\sigma_v^2+\psi_i)^{-1}(\sigma_v^2+\psi_i)}_{=1}\left(\mathbf{z_i z_i^T}\right)^{-1}\mathbf{z_i^T z}_i$$

$$-\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2\underbrace{(\sigma_v^2+\psi_i)^{-1}(\sigma_v^2+\psi_i)}_{=1}+\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2$$

$$= \mathbf{z_i^T z}_i\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2\left(\mathbf{z_i z_i^T}\right)^{-1}\mathbf{z_i^T z}_i\underbrace{-\mathbf{z_i^T z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2+\mathbf{z_i^T z}_i\left(\mathbf{z_i z_i^T}\right)^{-1}\sigma_v^2}_{=0}$$

$$= \sigma_v^2\,\frac{\mathbf{z_i^T}}{\mathbf{z_i z_i^T}}\underbrace{\frac{\mathbf{z_i z_i^T}}{\mathbf{z_i z_i^T}}}_{=1}\mathbf{z}_i$$

$$= \sigma_v^2\,\frac{\mathbf{z_i^T z}_i}{\mathbf{z_i z_i^T}}$$

$$= \sigma_v^2\left(\mathbf{z_i z_i^T}\right)^{-1}\mathbf{z_i^T z_i}$$

$$III.\, E\left[(v_i+e_i)\left(\widehat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)^T\mathbf{z}_i\right]$$

$$= E\left[(v_i+e_i)\left(\mathbf{z_i}\left(\mathbf{z_i z_i^T}\right)^{-1}(v_i+e_i)\right)^T\mathbf{z_i}\right]$$

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$$= E\left[(v_i + e_i)(v_i + e_i)^T \left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i\right]$$

$$= E[(v_i + e_i)(v_i + e_i)^T]\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i$$

$$= (\sigma_v^2 + \psi_i)\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i$$

$$IV. E\left[(v_i + e_i)\left(\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\hat{\theta}_i - \mathbf{z}_\mathbf{i}^\mathbf{T} \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) - v_i\right)^T\right]$$

$$= E\left[(v_i + e_i)\left(\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\mathbf{z}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta} + v_i + e_i - \mathbf{z}_\mathbf{i}^\mathbf{T} \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) - v_i\right)^T\right]$$

$$= E\left[(v_i + e_i)\left(\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right) + \sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i) - v_i\right)^T\right]$$

$$= E\left[(v_i + e_i)\left(\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)^T \mathbf{z}_i + \sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)^T - {v_i}^T\right)\right]$$

$$= E\begin{bmatrix} \sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\sigma_v^2)\right)^T \mathbf{z}_i \\ +\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)(v_i + e_i)^T - (v_i + e_i){v_i}^T \end{bmatrix}$$

$$= E\begin{bmatrix} \sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)\left(-\mathbf{z}_i\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}(v_i + e_i)\right)^T \mathbf{z}_i \\ +\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)(v_i + e_i)^T - (v_i + e_i){v_i}^T \end{bmatrix}$$

$$= E\begin{bmatrix} -\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)(v_i + e_i)^T\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i \\ +\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)(v_i + e_i)^T - (v_i + e_i){v_i}^T \end{bmatrix}$$

$$= -E\left[\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)(v_i + e_i)^T\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i\right]$$
$$+E[\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(v_i + e_i)(v_i + e_i)^T] - E[(v_i + e_i){v_i}^T]$$

$$= -\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}E[(v_i + e_i)(v_i + e_i)^T]\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i$$
$$+\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}E[(v_i + e_i)(v_i + e_i)^T] - E[(v_i{v_i}^T) + ({v_i}^T e_i)]$$

$$= -\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(E[v_i{v_i}^T] + E[v_i{e_i}^T] + E[e_i{v_i}^T] + E[e_i{e_i}^T])\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i$$
$$+\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}(E[v_i{v_i}^T] + E[v_i{e_i}^T] + E[e_i{v_i}^T] + E[e_i{e_i}^T])$$
$$-E[(v_i{v_i}^T) + ({v_i}^T e_i)]$$

$$= -\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\sigma_v^2 + 0 + 0 + \psi_i\right)\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i$$
$$+\sigma_v^2(\sigma_v^2 + \psi_i)^{-1}\left(\sigma_v^2 + 0 + 0 + \psi_i\right) - E[(v_i{v_i}^T) + ({v_i}^T e_i)]$$

$$= -\sigma_v^2 \underbrace{(\sigma_v^2 + \psi_i)^{-1}(\sigma_v^2 + \psi_i)}_{=1}\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i + \sigma_v^2 \underbrace{(\sigma_v^2 + \psi_i)^{-1}(\sigma_v^2 + \psi_i)}_{=1} - (\sigma_v^2 + 0)$$

$$= -\sigma_v^2\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i + \sigma_v^2 - \sigma_v^2$$

$$= -\sigma_v^2\left(\mathbf{z}_i \mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1} \mathbf{z}_\mathbf{i}^\mathbf{T} \mathbf{z}_i$$

Lampiran 12 (lanjutan)

Berdasarkan $I, II, III,$ dan IV diperoleh:

$$cov\left\{\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right], \left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]\right\}$$

$$= cov\left\{\left[\hat{\theta}_i - \mathbf{z}_\mathbf{i}^\mathbf{T}\widehat{\boldsymbol{\beta}}(\sigma_v^2)\right], \left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]\right\}$$

$$= -(\sigma_v^2 + \psi_i)\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}} + \sigma_v^2\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}$$

$$+(\sigma_v^2 + \psi_i)\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}} - \sigma_v^2\left(\mathbf{z}_{\boldsymbol{i}}\mathbf{z}_\mathbf{i}^\mathbf{T}\right)^{-1}\mathbf{z}_\mathbf{i}^\mathbf{T}\mathbf{z}_{\boldsymbol{i}}$$

$$= 0.$$

Karena $cov\left\{\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right], \left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]\right\} = 0$, maka $\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]$ dan $\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2) - \theta_i\right]$ independen, sehingga:

$$\mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2})\right]$$

$$= \mathrm{MSE}\left[\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right] + E\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2.$$

Lampiran 13. Bukti $E\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2 = g_3(\sigma_v^2)$

Akan ditunjukkan bahwa:

$$E\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^{\mathbf{2}} \approx \frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}[\boldsymbol{\mathcal{L}}(\sigma_{\boldsymbol{v}}^{\mathbf{2}})]^{-1}$$

$$= \frac{2{\psi_i}^2}{(\sigma_{\boldsymbol{v}}^{\mathbf{2}}+\psi_i)}$$

$$= g_3(\sigma_{\boldsymbol{v}}^{\mathbf{2}})$$

Dengan menggunakan aproksimasi Taylor orde pertama dari $\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2})$ di sekitar σ_v^2.

$$\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) \approx \left(T(\hat{\theta}_i)\right)(\sigma_v^2) + \frac{\partial\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)}{\partial\sigma_v^2}\left(\widehat{\sigma_v^2}-\sigma_v^2\right)$$

$$\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2) \approx \frac{\partial\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)}{\partial\sigma_v^2}\left(\widehat{\sigma_v^2}-\sigma_v^2\right)$$

$$\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_v^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2 \approx \left[\frac{\partial\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)}{\partial\sigma_v^2}\left(\widehat{\sigma_v^2}-\sigma_v^2\right)\right]^2$$

$$\approx \left[\left(\frac{\partial\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)}{\partial\sigma_v^2}\right)^T\left(\widehat{\sigma_v^2}-\sigma_v^2\right)\right]^2$$

Berdasarkan persamaan (3.13):

$$\left(T(\hat{\theta}_i)\right)(\sigma_v^2) = \mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) + \left(1-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)$$

$$\frac{\partial\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)}{\partial\sigma_v^2} \approx \frac{\partial\left(\mathbf{z_i^T}\boldsymbol{\beta} + \frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right) + \left(1-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)}{\partial\sigma_v^2}$$

$$\approx \underbrace{\frac{\partial(\mathbf{z_i^T}\boldsymbol{\beta})}{\partial\sigma_v^2}}_{=0} + \frac{\partial\left(\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\left(\hat{\theta}_i - \mathbf{z_i^T}\boldsymbol{\beta}\right)\right)}{\partial\sigma_v^2}$$

$$+\underbrace{\frac{\partial\left(\left(1-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)\mathbf{z_i^T}\left(\hat{\boldsymbol{\beta}}(\sigma_v^2)-\boldsymbol{\beta}\right)\right)}{\partial\sigma_v^2}}_{=0}$$

Lampiran 13 (lanjutan)

dengan asumsi $\frac{\partial\left(\left(1-\frac{\sigma_v^2}{(\sigma_v^2+\psi_i)}\right)\mathbf{z}_i^T\right)}{\partial\sigma_v^2} \approx 0$, maka diperoleh:

$$\frac{\partial\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)}{\partial\sigma_v^2} \approx \frac{\partial\left(\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}(\hat{\theta}_i - \mathbf{z}_i^T\boldsymbol{\beta})\right)}{\partial\sigma_v^2}$$

sehingga

$$\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2 \approx \left[\left(\frac{\partial\left(\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right)}{\partial\sigma_v^2}\right)^T\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\right]^2$$

$$\approx \left[\left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}(\hat{\theta}_i - \mathbf{z}_i^T\boldsymbol{\beta})\right)^T\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\right]^2$$

Kemudian dengan mengambil nilai ekspektasinya, didapat:

$$E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right) - \left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^2$$

$$\approx E\left[\left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}(\hat{\theta}_i - \mathbf{z}_i^T\boldsymbol{\beta})\right)^T\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\right]^2$$

$$\approx E\left[\left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}(v_i+e_i)\right)\left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}(v_i+e_i)\right)^T\right]E\left[\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T\right]$$

$$\approx E\left[\left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}\right)(v_i+e_i)(v_i+e_i)^T\left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}\right)^T\right]\underbrace{E\left[\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T\right]}_{=\left[\mathcal{L}_{\sigma_v^2}(\sigma_v^2)\right]^{-1}}$$

$$\approx \left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}\right)E\left[(v_i+e_i)(e_i{}^T+v_i{}^T)\right]\left(\frac{\partial\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial\sigma_v^2}\right)^T\left[\mathcal{L}_{\sigma_v^2}(\sigma_v^2)\right]^{-1}$$

Lampiran 13 (lanjutan)

$$\approx \left(\frac{\partial \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial \sigma_v^2}\right) E[v_i e_i^T + v_i v_i^T + e_i e_i^T + e_i v_i^T] \left(\frac{\partial \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial \sigma_v^2}\right)^T [\boldsymbol{\mathcal{L}}_{\sigma_{\boldsymbol{v}}^2}(\sigma_{\boldsymbol{v}}^{\mathbf{2}})]^{-1}$$

$$\approx \left(\frac{\partial \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial \sigma_v^2}\right) (0 + \sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i + 0) \left(\frac{\partial \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial \sigma_v^2}\right)^T [\boldsymbol{\mathcal{L}}_{\sigma_{\boldsymbol{v}}^2}(\sigma_{\boldsymbol{v}}^{\mathbf{2}})]^{-1}$$

$$\approx \left(\frac{\partial \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial \sigma_v^2}\right) (\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i) \left(\frac{\partial \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial \sigma_v^2}\right)^T [\boldsymbol{\mathcal{L}}_{\sigma_{\boldsymbol{v}}^2}(\sigma_{\boldsymbol{v}}^{\mathbf{2}})]^{-1}$$

dengan

$$\begin{aligned}
\frac{\partial \frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}}{\partial \sigma_v^2} &= \frac{\partial}{\partial \sigma_v^2}\left(\frac{\sigma_v^2}{\left(\sigma_v^2+\psi_i\right)}\right) \\
&= \frac{1.\left(\sigma_v^2+\psi_i\right) - \sigma_v^2 . 1}{\left(\sigma_v^2+\psi_i\right)^2} \\
&= \frac{\psi_i}{\left(\sigma_v^2+\psi_i\right)^2}
\end{aligned}$$

maka didapat:

$$E\left[\left(T(\hat{\theta}_i)\right)(\widehat{\sigma_{\boldsymbol{v}}^2}) - \left(T(\hat{\theta}_i)\right)(\sigma_{\boldsymbol{v}}^2)\right]^{\mathbf{2}}$$

$$\approx \left(\frac{\psi_i}{(\sigma_v^2+\psi_i)^2}\right)(\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i)\left(\frac{\psi_i}{(\sigma_v^2+\psi_i)^2}\right)^T [\boldsymbol{\mathcal{L}}_{\sigma_{\boldsymbol{v}}^2}(\sigma_{\boldsymbol{v}}^{\mathbf{2}})]^{-1}.$$

dimana

$$\begin{aligned}
[\boldsymbol{\mathcal{L}}_{\sigma_{\boldsymbol{v}}^2}(\sigma_{\boldsymbol{v}}^{\mathbf{2}})]^{-1} &= var(\sigma_{\boldsymbol{v}}^{\mathbf{2}}) \\
&= \left[E\left(\frac{\partial^2 \ln L(\hat{\boldsymbol{\beta}}, \sigma_{\boldsymbol{v}}^{\mathbf{2}}; \hat{\theta}_i)}{\partial \sigma_{\boldsymbol{v}}^{\mathbf{2}} \, \partial \sigma_{\boldsymbol{v}}^{\mathbf{2}}}\right)\right]^{-1} \\
&= \left[\frac{1}{2}\frac{1}{\left(\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i\right)^2}\right]^{-1} \quad \text{; berdasarkan persamaan (3.18)} \\
&= 2(\sigma_{\boldsymbol{v}}^{\mathbf{2}} + \psi_i)^2.
\end{aligned}$$

Lampiran 13 (lanjutan)

Dengan demikian,

$$E\left[\left(T(\hat{\theta}_i)\right)\left(\widehat{\sigma_v^2}\right)-\left(T(\hat{\theta}_i)\right)(\sigma_v^2)\right]^{\mathbf{2}} \approx \frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}\left[\mathcal{L}(\sigma_v^2)\right]^{-1}$$

$$= \frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}\left[2(\sigma_v^2+\psi_i)^2\right]$$

$$= \frac{2{\psi_i}^2}{(\sigma_v^2+\psi_i)}$$

$$= g_3(\sigma_v^2) \qquad \text{(terbukti).}$$

Lampiran 14. Bukti $E\left[g_1\left(\widehat{\sigma_v^2}\right)\right] \approx g_1(\sigma_v^2) + c_{\widehat{\sigma_v^2}}^T(\sigma_v^2)\nabla g_1(\sigma_v^2) - g_3(\sigma_v^2)$, $E\left[g_2\left(\widehat{\sigma_v^2}\right)\right] \approx g_2(\sigma_v^2)$, dan $E\left[g_3\left(\widehat{\sigma_v^2}\right)\right] \approx g_3(\sigma_v^2)$

a. Akan ditunjukkan bahwa:

$E\left[g_1\left(\widehat{\sigma_v^2}\right)\right] \approx g_1(\sigma_v^2) + c_{\widehat{\sigma_v^2}}^T(\sigma_v^2)\nabla g_1(\sigma_v^2) - g_3(\sigma_v^2)$

Dengan menggunakan ekspansi Taylor untuk $g_1(\sigma_v^2)$ di sekitar σ_v^2, didlapat:

$$g_1\left(\widehat{\sigma_v^2}\right) = P_2\left(\widehat{\sigma_v^2}\right) + R_3\left(\widehat{\sigma_v^2}\right)$$

$$= g_1(\sigma_v^2) + \left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla g_1(\sigma_v^2) + \frac{1}{2}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla^2 g_1(\sigma_v^2)\left(\widehat{\sigma_v^2} - \sigma_v^2\right)$$

$$+R_3\left(\widehat{\sigma_v^2}\right)$$

dimana

$\nabla g_1(\sigma_v^2)$ adalah turunan pertama dari $g_1(\sigma_v^2) = \frac{\sigma_v^2\psi_i}{(\sigma_v^2+\psi_i)}$ terhadap σ_v^2.

$$\frac{\partial}{\partial\sigma_v^2}\, g_1(\sigma_v^2) = \frac{\partial}{\partial\sigma_v^2}\left(\frac{\sigma_v^2\psi_i}{(\sigma_v^2 + \psi_i)}\right)$$

$$= \frac{\psi_i(\sigma_v^2 + \psi_i) - \sigma_v^2\psi_i}{(\sigma_v^2 + \psi_i)^2}$$

$$= \frac{{\psi_i}^2}{(\sigma_v^2 + \psi_i)^2}$$

$\nabla^2 g_1(\sigma_v^2)$ adalah turunan kedua dari $g_1(\sigma_v^2)$ terhadap σ_v^2.

$$\frac{\partial^2}{\partial\sigma_v^2\partial\sigma_v^2}\, g_1(\sigma_v^2) = \frac{\partial}{\partial\sigma_v^2}\left(\frac{{\psi_i}^2}{(\sigma_v^2 + \psi_i)^2}\right)$$

$$= \frac{-2{\psi_i}^2}{(\sigma_v^2 + \psi_i)^3}$$

dan

$$R_3\left(\widehat{\sigma_v^2}\right) = \frac{1}{3!}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^3 \nabla^3 g_1\left(\xi\left(\widehat{\sigma_v^2}\right)\right)$$

$$= G\left(\widehat{\sigma_v^2} - \sigma_v^2\right)$$

ambil $H\left(\widehat{\sigma_v^2} - \sigma_v^2\right) = \left(\widehat{\sigma_v^2} - \sigma_v^2\right)$ maka dapat dihitung nilai limit dari:

$$\lim_{\left(\widehat{\sigma_v^2}-\sigma_v^2\right)\to\sim}\left|\frac{G\left(\widehat{\sigma_v^2} - \sigma_v^2\right)}{H\left(\widehat{\sigma_v^2} - \sigma_v^2\right)}\right| = \lim_{\left(\widehat{\sigma_v^2}-\sigma_v^2\right)\to\sim}\left|\frac{\frac{1}{3!}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^3 \nabla^3 g_1\left(\xi\left(\widehat{\sigma_v^2}\right)\right)}{\left(\widehat{\sigma_v^2} - \sigma_v^2\right)}\right| = O\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^3$$

Lampiran 14 (lanjutan)

sehingga diperoleh:

$$g_1\left(\widehat{\sigma_v^2}\right) = g_1(\sigma_v^2) + \left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla g_1(\sigma_v^2) + \frac{1}{2}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla^2 g_1(\sigma_v^2)\left(\widehat{\sigma_v^2} - \sigma_v^2\right)$$
$$+O\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^3$$
$$\approx g_1(\sigma_v^2) + \left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla g_1(\sigma_v^2) + \frac{1}{2}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla^2 g_1(\sigma_v^2)\left(\widehat{\sigma_v^2} - \sigma_v^2\right)$$

$$E\left[g_1\left(\widehat{\sigma_v^2}\right)\right] \approx E\left[g_1(\sigma_v^2) + \left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla g_1(\sigma_v^2) + \frac{1}{2}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \nabla^2 g_1(\sigma_v^2)\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\right]$$
$$\approx E\begin{bmatrix} g_1(\sigma_v^2) + \left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \dfrac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^2} \\ +\dfrac{1}{2}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T \dfrac{-2{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\end{bmatrix}$$
$$\approx E[g_1(\sigma_v^2)] + \frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^2}\underbrace{E\left[\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T\right]}_{c^T_{\widehat{\sigma_v^2}}(\sigma_v^2)}$$
$$+\frac{1}{2}\frac{-2{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}\underbrace{E\left[\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\right]}_{\mathcal{L}^{-1}_{\sigma_v^2}(\sigma_v^2)}$$
$$\approx g_1(\sigma_v^2) + c^T_{\widehat{\sigma_v^2}}(\sigma_v^2)\left(\frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^2}\right)$$
$$-\frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}\underbrace{E\left[\left(\widehat{\sigma_v^2} - \sigma_v^2\right)^T\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\right]}_{\mathcal{L}^{-1}_{\sigma_v^2}(\sigma_v^2)}$$

Berdasarkan Lampiran 13, diketahui bahwa $g_3(\sigma_v^2) = \frac{2{\psi_i}^2}{(\sigma_v^2+\psi_i)}$. Bentuk $\frac{2{\psi_i}^2}{(\sigma_v^2+\psi_i)}$ ini dapat dituliskan sebagai berikut:

$$g_3(\sigma_v^2) = \frac{2{\psi_i}^2}{(\sigma_v^2+\psi_i)}$$
$$= \frac{2{\psi_i}^2}{(\sigma_v^2+\psi_i)} \cdot \frac{(\sigma_v^2+\psi_i)^2}{(\sigma_v^2+\psi_i)^2}$$
$$= \frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}\left[2(\sigma_v^2+\psi_i)^2\right]$$
$$= \left(\frac{{\psi_i}^2}{(\sigma_v^2+\psi_i)^3}\right)\mathcal{L}^{-1}_{\sigma_v^2}(\sigma_v^2).$$

Lampiran 14 (lanjutan)

Dengan demikian, didapat bentuk $E\left[g_1\left(\widehat{\sigma_v^2}\right)\right]$ berikut:

$E\left[g_1\left(\widehat{\sigma_v^2}\right)\right] \approx g_1(\sigma_v^2) + c_{\widehat{\sigma_v^2}}^T(\sigma_v^2)\nabla g_1(\sigma_v^2) - g_3(\sigma_v^2).$

b. Akan ditunjukkan bahwa:

$$E\left[g_2\left(\widehat{\sigma_v^2}\right)\right] \approx g_2(\sigma_v^2)$$

Dengan menggunakan ekspansi Taylor untuk $g_2(\sigma_v^2)$ di sekitar σ_v^2 didapat:

$$\begin{aligned} g_2\left(\widehat{\sigma_v^2}\right) &= P_0\left(\widehat{\sigma_v^2}\right) + R_1\left(\widehat{\sigma_v^2}\right) \\ &= g_2(\sigma_v^2) + R_1\left(\widehat{\sigma_v^2}\right) \end{aligned}$$

dimana

$$\begin{aligned} R_1\left(\widehat{\sigma_v^2}\right) &= \frac{1}{1!}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\nabla g_2\left(\xi\left(\widehat{\sigma_v^2}\right)\right) \\ &= I\left(\widehat{\sigma_v^2} - \sigma_v^2\right) \end{aligned}$$

ambil $H\left(\widehat{\sigma_v^2} - \sigma_v^2\right) = \left(\widehat{\sigma_v^2} - \sigma_v^2\right)$ maka dapat dihitung nilai limit dari:

$$\begin{aligned} \lim_{\left(\widehat{\sigma_v^2} - \sigma_v^2\right) \to \sim} \left|\frac{I\left(\widehat{\sigma_v^2} - \sigma_v^2\right)}{H\left(\widehat{\sigma_v^2} - \sigma_v^2\right)}\right| &= \lim_{\left(\widehat{\sigma_v^2} - \sigma_v^2\right) \to \sim} \left|\frac{\frac{1}{1!}\left(\widehat{\sigma_v^2} - \sigma_v^2\right)\nabla g_2\left(\xi\left(\widehat{\sigma_v^2}\right)\right)}{\left(\widehat{\sigma_v^2} - \sigma_v^2\right)}\right| \\ &= O\left(\widehat{\sigma_v^2} - \sigma_v^2\right) \end{aligned}$$

sehingga diperoleh:

$$\begin{aligned} g_2\left(\widehat{\sigma_v^2}\right) &= g_2(\sigma_v^2) + O\left(\widehat{\sigma_v^2} - \sigma_v^2\right) \\ &\approx g_2(\sigma_v^2) \end{aligned}$$

dengan mengambil nilai ekspektasinya, diperoleh:

$$\begin{aligned} E\left[g_2\left(\widehat{\sigma_v^2}\right)\right] &\approx E[g_2(\sigma_v^2)] \\ &\approx g_2(\sigma_v^2) \end{aligned}$$

Jadi, $E\left[g_2\left(\widehat{\sigma_v^2}\right)\right] \approx g_2(\sigma_v^2)$. (terbukti)

c. Akan ditunjukkan bahwa:

$$E\left[g_3\left(\widehat{\sigma_v^2}\right)\right] \approx g_3(\sigma_v^2)$$

Lampiran 14 (lanjutan)

Dengan menggunakan ekspansi Taylor untuk $g_3(\sigma_v^2)$ di sekitar σ_v^2 didapat:

$$\begin{aligned} g_3(\widehat{\sigma_v^2}) &= P_0(\widehat{\sigma_v^2}) + R_1(\widehat{\sigma_v^2}) \\ &= g_3(\sigma_v^2) + R_1(\widehat{\sigma_v^2}) \end{aligned}$$

dimana

$$\begin{aligned} R_1(\widehat{\sigma_v^2}) &= \frac{1}{1!}(\widehat{\sigma_v^2} - \sigma_v^2)\, \nabla g_3\left(\xi(\widehat{\sigma_v^2})\right) \\ &= K(\widehat{\sigma_v^2} - \sigma_v^2) \end{aligned}$$

ambil $H(\widehat{\sigma_v^2} - \sigma_v^2) = (\widehat{\sigma_v^2} - \sigma_v^2)$ maka dapat dihitung nilai limit dari:

$$\begin{aligned} \lim_{(\widehat{\sigma_v^2} - \sigma_v^2) \to \sim} \left| \frac{I(\widehat{\sigma_v^2} - \sigma_v^2)}{K(\widehat{\sigma_v^2} - \sigma_v^2)} \right| &= \lim_{(\widehat{\sigma_v^2} - \sigma_v^2) \to \sim} \left| \frac{\frac{1}{1!}(\widehat{\sigma_v^2} - \sigma_v^2)\, \nabla g_3\left(\xi(\widehat{\sigma_v^2})\right)}{(\widehat{\sigma_v^2} - \sigma_v^2)} \right| \\ &= O(\widehat{\sigma_v^2} - \sigma_v^2) \end{aligned}$$

sehingga diperoleh:

$$\begin{aligned} g_3(\widehat{\sigma_v^2}) &= g_3(\sigma_v^2) + O(\widehat{\sigma_v^2} - \sigma_v^2) \\ &\approx g_3(\sigma_v^2) \end{aligned}$$

dengan mengambil nilai ekspektasinya, diperoleh:

$$\begin{aligned} E\left[g_3(\widehat{\sigma_v^2})\right] &\approx E[g_3(\sigma_v^2)] \\ &\approx g_3(\sigma_v^2) \end{aligned}$$

Jadi, $E\left[g_3(\widehat{\sigma_v^2})\right] \approx g_3(\sigma_v^2)$. (terbukti)

Lampiran 15. Uji Kenormalan Taksiran Langsung

Uji Kolmogorov-Smirnov

Hipotesis:

H_0: nilai taksiran langsung berdistribusi normal

H_1: tidak demikian

Tingkat Signifikansi: $\alpha = 0.05$

Dengan menggunakan *software* SPSS 16.0, akan dilakukan uji Kolmogorov-Smirnov, diperoleh output sebagai berikut.

Tests of Normality

	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
taksiran_langsung	.102	35	.200*	.972	35	.503

a. Lilliefors Significance Correction

*. This is a lower bound of the true significance.

Berdasarkan output tersebut, diperoleh $\hat{\alpha} = 0.200 > \alpha = 0.05$, maka dengan tingkat kepercayaan 95%, H_0 tidak ditolak. Jadi, nilai taksiran langsung pengeluaran rumah tangga per kapita berdistribusi normal.

Lampiran 16. Program EBLUP ML.r pada Software R.

```
ydirect=read.table(file="D:/ydirect.txt")
Zpop=read.table(file="D:/dataZ.txt")
vardirect= read.table(file="D:/vardirect.txt")
ydirect=as.matrix(ydirect)
Zpop=as.matrix(Zpop)
vardirect=as.matrix(vardirect)
m=35

#EBLUB dengan metode ML, scoring algorithm
EBLUP.area.ML <- function(ydirect, Zpop, vardirect, m,
tol=10e-5, maxiter=50)
{
    sigma2.v.estimate <- 0  #komponen variansi
    sigma2.v.estimate[1] <- 5000000 #inisial value
    k <- 0
    diff <- tol+1
    while ((diff>tol) & (k<maxiter))
    {
         k <- k+1
         V <- diag(1,m) #matriks varians-kovarians

         for (i in 1:m)
         {
              V[i,i] <-
(sigma2.v.estimate[k]+vardirect[i,1])
         }
         #vardirect merupakan vektor variansi sampel,
berukuran mx1
         Vinvers <- solve(V)
         Beta <-
         solve(t(Zpop)%*%Vinvers%*%Zpop)%*%t(Zpop)%*%Vinve
         rs%
         *%ydirect[,1] #ydirect merupakan vektor dari
         taksiran langsung,berukuran mx1
         sdev <-(-0.5)*sum(diag(Vinvers))-
         (0.5)*t(ydirect[,1]-(Zpop)%*%Beta)%*%((-
         1)*Vinvers%*%Vinvers)%*%(ydirect[,1]-
         (Zpop)%*%Beta) #Xpop adalah matriks dari p
         auxiliary variables, berukuran mxp
         Idev <-((0.5)*(sum(diag(Vinvers%*%Vinvers))))^(-
         1) #Idev is invers dari matriks informasi
         #scoring algorithm
         sigma2.v.estimate[k+1] <-
         sigma2.v.estimate[k]+Idev*sdev
         diff <- abs(sigma2.v.estimate[k+1]-
         sigma2.v.estimate[k]) #batas toleransi
    }

    if (sigma2.v.estimate[k+1]<0)
         sigma2v <- 0
```

Lampiran 16 (lanjutan)

```
else
      sigma2v <- sigma2.v.estimate[k+1]

V <- diag(1,m)
G <- diag(1,m)*sigma2v

for (i in 1:m)
{
      V[i,i] <-(sigma2v+vardirect[i,1])
}
Vinvers <- solve(V)
Bestimate <-
solve(t(Zpop)%*%Vinvers%*%Zpop)%*%t(Zpop)%*%Vinvers%*%
ydirect[,1]
m1 <- diag(1,m)
thetaEBLUP <-
Zpop%*%Bestimate+m1%*%G%*%Vinvers%*%(ydirect[,1]-
(Zpop%*%Bestimate)) #EBLUP estimator
efekrandom <- m1%*%G%*%Vinvers%*%(ydirect[,1]-
(Zpop%*%Bestimate))#Taksiran efek random

#MSE EBLUP small area
#g1
g1 <- diag((G-G%*%Vinvers%*%G))
#g2
g2 <- matrix(0,m,1)
for (i in 1:m)
{
      m1 <- matrix(0,m,1)
      m1[i] <- 1
      g2[i] <-(Zpop[i,]-
      t(m1)%*%G%*%Vinvers%*%Zpop)%*%solve(t(Zpop)%*%Vin
      vers%*%Zpop)%*%t(Zpop[i,]-
      t(m1)%*%G%*%Vinvers%*%Zpop)
}
Idev <-((0.5)*(sum(diag(Vinvers%*%Vinvers))))^(-1)
#g3
g3 <- matrix(0,m,1)
for (i in 1:m)
{     m1 <- matrix(0,m,1)
      m1[i] <- 1
      g3[i] <-(t(m1)%*%(Vinvers+G%*%((-
      1)*Vinvers%*%Vinvers))%*%V%*%t((t(m1)%*%(Vinvers+
      G%*%((-1)*Vinvers%*%Vinvers)))))*Idev
}

#bias
bdist <- 0
btr <- 0
gradg1 <- 0
I <- diag(1,m)
```

Lampiran 16 (lanjutan)

```
     distorsione <- matrix(0,m,1)
     for (i in 1:m)
     {
          m1 <- matrix(0,m,1)
          m1[i] <- 1
          gradg1 <- t(m1)%*%(I-((I%*%Vinvers%*%G)+(G%*%((-
          1)*Vinvers%*%I%*%Vinvers)%*%G)+(G%*%Vinvers%*%I))
          )%*%m1
          btr <-
          sum(diag(solve(t(Zpop)%*%Vinvers%*%Zpop)%*%t(Zpop
          )%*%((-1)*Vinvers%*%I%*%Vinvers)%*%Zpop))
          bdist <- (1/(m*2))*Idev*btr
          distorsione[i,1] <- bdist*gradg1
     }

     #Taksiran MSE EBLUP
     mseestimate <- g1-distorsione+g2+2*g3
     #list(beta=Bestimate, EBLUP=thetaEBLUP, g1=g1, g2=g2,
g3=g3,
     #MSE=mseestimate)
     list(EBLUP=thetaEBLUP, beta=Bestimate,
     sigma2v=sigma2.v.estimate[k+1],
     g1=g1, g2=g2, g3=g3, mse=mseestimate,
efekrandom=efekrandom)
}

EBLUP.area.ML(ydirect, Zpop, vardirect, m, tol=10e-5,
maxiter=50)
```

Lampiran 17. Uji Kenormalan Taksiran Pengaruh Acak *Small Area*

Uji Kolmogorov-Smirnov

Hipotesis:

H_0: nilai taksiran pengaruh acak berdistribusi normal

H_1: tidak demikian

Tingkat Signifikansi: $\alpha = 0.05$

Dengan menggunakan *software* SPSS 16.0, akan dilakukan uji Kolmogorov-Smirnov, diperoleh output sebagai berikut.

Tests of Normality

	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
pengaruh_acak	.086	35	.200*	.982	35	.820

a. Lilliefors Significance Correction

*. This is a lower bound of the true significance.

Berdasarkan output tersebut, diperoleh $\hat{\alpha} = 0.200 > \alpha = 0.05$, maka dengan tingkat kepercayaan 95%, H_0 tidak ditolak. Jadi, nilai taksiran pengaruh acak berdistribusi normal.