TRAJECTORY SHAPING OF SURFACE-TO-SURFACE MISSILE WITH TERMINAL IMPACT ANGLE CONSTRAINT

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Abstract

This paper presents trajectory shaping of a surface-to-surface missile attacking a fixed with terminal impact angle constraint. The missile must hit the target from above, subject to the missile dynamics and path constraints. The problem is reinterpreted using optimal control theory resulting in the formulation of minimum integrated altitude. The formulation entails nonlinear, two-dimensional missile flight dynamics, boundary conditions and path constraints. The generic shape of optimal trajectory is: level flight, climbing, diving; this combination of the three flight phases is called the bunt manoeuvre. The numerical solution of optimal control problem is solved by a direct collocation method. The computational results is used to reveal the structure of optimal solution which is composed of several arcs, each of which can be identified by the corresponding manoeuvre executed and constraints active.

Keywords: trajectory shaping, direct collocation, optimal control, minimum integrated altitude

1. Introduction

Trajectory shaping of a missile is an advanced approach to missile guidance which aims at computing the whole trajectory in an optimal way. Trajectory shaping has been used by some researchers to improve the precision of the missile trajectory during manoeuvre. To increase warhead effectiveness and survivability again missile defence system the optimization of terminal impact angle is necessary to modulate trajectories of the missile. Furthermore it can be use to enhance the overall productivity and the effectiveness of the operation, especially to avoid from increasing anti-missile capability [1].

Ryoo, Cho and Tahk [2] have studied an optimal guidance law for constant speed missile with constraint on the impact angle and control input. Optimal guidance law with weights on the terminal velocity has been investigated by Ben-Asher and Yaesh [3]. The computational analysis for the terminal bunt manoeuvre for the minimum altitude and time-optimal along the optimal trajectory for surface to surface missile guidance based on the Pontryagin's Minimum Principle was given by Subchan et al. [4,5].

This paper presents some computational results of the optimal trajectory of missile which minimising the integrated altitude along the optimal trajectory. Furthermore, we investigate the missile trajectory by varying the terminal impact engagement. The essence of approach is to compute an optimal trajectory together with the associated control demand. In other words, for given launch and strike conditions, find a missile trajectory which hits the target in a pre-defined way and shapes the missile's flight in an optimal fashion. This setting leads naturally to expressing the guidance problem as an optimal control problem. Hence the solution approach for the trajectory shaping involves computational optimal control.

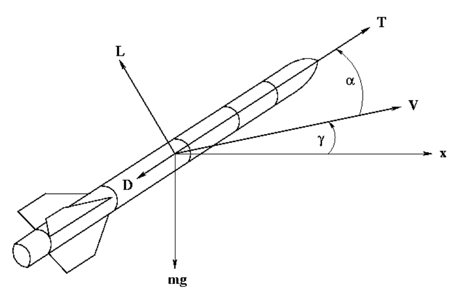

Figure 1. **Definition of the missile axes and angles. Note that *L* is the normal aerodynamic force and *D* is the axial aerodynamic force with respect to a body-axis frame, not lift and drag**

The dynamic equations of a point mass missile moving in the vertical plane over flat non-rotating earth can be given as follows

$$\dot{\gamma} = \frac{T-D}{mV}\sin\alpha + \frac{L}{mV}\cos\alpha - \frac{g\cos\gamma}{V} \quad (1a)$$

$$\dot{V} = \frac{T-D}{m}\cos\alpha - \frac{L}{m}\sin\alpha - g\sin\gamma \quad (1b)$$

$$\dot{x} = V\cos\gamma \quad (1c)$$

$$\dot{h} = V\sin\gamma\,, \quad (1d)$$

where t is the actual time, $t_0 \le t \le t_f$ with t_0 as the initial time and t_f as the final time. The state variables are the flight path angle γ, speed V, horizontal position x and altitude h of the missile. The thrust magnitude T and the angle of attack α are the two control variables (see Figure 1). The aerodynamic forces D and L are functions of the altitude h, velocity V and angle of attack α. The following relationships have been assumed:

Axial aerodynamic force. The drag D is written in the form

$$D(h,V,\alpha) = \frac{1}{2}C_d\rho V^2 S_{ref} \quad (2)$$

$$C_d = A_1\alpha^2 + A_2\alpha + A_3. \quad (3)$$

Note that D is not the drag force.

Normal aerodynamic force. The lift L is written in the form

$$L(h,V,\alpha) = \frac{1}{2}C_l\rho V^2 S_{ref} \quad (4)$$

$$C_l = B_1\alpha + B_2\,, \quad (5)$$

where ρ is air density given by

$$\rho = C_1h^2 + C_2h + C_3\,, \quad (6)$$

and S_{ref} is the reference area of the missile; m denotes the mass and g the gravitational constant, see also Table 1. Note that L is not the drag force.

Table 1. Physical modelling parameters

Quantity	Value	Unit
m	1005	Kg
g	9.81	m/s²
S_{ref}	0.3376	m²
A_1	-1.9431	
A_2	-0.1499	
A_3	0.2359	
B_1	21.9	
B_2	0	
C_1	$3.312\ 10^{-9}$	kg m^{-5}
C_2	$-1.142\ 10^{-4}$	kg m^{-4}
C_3	1.224	kg m^{-3}

Table 2. Boundary conditions and constraints

Quantity	Value	Unit
V_{min}	200	m/s
V_{max}	310	m/s
L_{min}	-4	g
L_{min}	4	g
h_{min}	30	m
T_{min}	1000	N
T_{max}	6000	N

Boundary conditions. The initial and final conditions for the four state variables are specified:

$$\begin{aligned} \gamma(0) &= \gamma_0 & \gamma(t_f) &= \gamma_f \\ V(0) &= V_0 & V(t_f) &= V_f \\ x(0) &= x_0 & x(t_f) &= x_f \\ h(0) &= h_0 & h(t_f) &= h_f \end{aligned} \quad (7)$$

In addition, constraints are defined as follows:

- State path constraints

$$V_{\min} \le V \le V_{\max} \quad (8)$$

$$h_{\min} \le h \quad (9)$$

- Control path constraint

$$T_{\min} \le T \le T_{\max} \quad (10)$$

- Mixed state and control constraint

$$L_{\min} \le \frac{L}{mg} \le L_{\max} \quad (11)$$

where $L_{\min}$ and $L_{\max}$ are normalized, see Table 2.

Performance Index. The mission is to hit a fixed target while minimising the missile exposure to anti-air defences. The problem is to find the trajectory of a generic cruise missile from the assigned initial state to a final state with the minimum integrated altitude along the trajectory. The objective can be formulated by introducing the performance criterion:

$$J = \int_{t_0}^{t_f} h\,dt \quad (12)$$

2. Method

Direct method approach is based on the discretisation of both the state and/or control variables. A variety of direct method has been developed and applied. Gradient algorithms were proposed by Kelley [6] and by Bryson and Denham [7]. Pytlak solved a state constrained optimal control problem using a gradient algorithm and applied it for some problems (see [8], [9]). Hargraves and Paris [10] reintroduced the direct transcription approach, by discretising the dynamic equations using a collocation method. A cubic polynomial is used to approximate the state variables and linear interpolation for the control variables. The collocation scheme was originally used by Dickmanns and Well [11] to solve.

TPBVPs. Seywald et al. introduced an approach based on the representation of the dynamical system in terms of differential inclusions. This method employs the concepts of hodograph space and attainable sets (see [12,13]). Direct transcriptions have been presented in detail by many researchers, e.g., Betts et al. [14,15], Enright and Conway [16], Herman [17], Tang and Conway [18], Ross and Fahroo [19], Elnagar et al. [20, 21].

The basic approach for solving optimal control problem by direct collocation approach is to transform the optimal control problem into sequence of nonlinear constrained optimisation problems by discretising of the state and control variables. The duration time of the optimal trajectory is divided into subinterval as follows:

$$t_0 = t_1 < t_2 < t_3 \ldots < t_k = t_f$$

The state and control variables at each node are $x_j = x(t_j)$ and $u_j = u(t_j)$ such that the state and control variables at the nodes are defined as nonlinear programming variables:

$$Y = [u(t_1), \ldots, u(t_k), x(t_1), \ldots, x(t_k)] \qquad (13)$$

The controls are chosen as piecewise linear interpolating functions between $u(t_j)$ and $u(t_{j+1})$ for $t_j \le t \le t_{j+1}$ as follows:

$$u_{app}(t) = u(t_j) + \frac{t - t_j}{t_{j+1} - t_j}[u(t_{j+1}) - u(t_j)] \qquad (14)$$

The value of the control variables at the centre is given by

$$u(t_{c,j}) = \frac{u(t_j) + u(t_{j+1})}{2} \qquad (15)$$

The piecewise linear interpolation is used to prepare for the possibility of discontinuous solutions in control. The state variable $x(t)$ is approximated by a continuously differentiable and piecewise Hermite-Simpson cubic polynomial between $x(t_j)$ and $x(t_{j+1})$ on the interval $t_j \le t \le t_{j+1}$ of length q_j:

$$x_{app}(t) = \sum_{r=0}^{3} c_r^j \frac{t - t_j}{p_j} \qquad (16)$$

$$c_0^j = x(t_j)$$

$$c_1^j = q_j f_j$$

$$c_2^j = -3x(t_j) - 2q_j f_j + 3x(t_{j+1}) - q_j f_{j+1}$$

$$c_3^j = 2x(t_j) + q_j f_j - 2(t_{j+1}) + q_j f_{j+1}$$

where:

$$f_j = f(x(t_j), u(t_j), t_j), q_j = t_{j+1} - t_j$$

$$t_j \le t \le t_{j+1}, \quad j = 1, 2, \ldots, k-1$$

The value of the state variables at the centre point of the cubic approximation

$$x_{c,j} = \frac{x(t_j) + x(t_{j+1})}{2} + \frac{q\left(f(t_j) + f(t_{j+1})\right)}{8} \qquad (17)$$

and the derivative is

$$\frac{dx_{c,j}}{dt} = -\frac{3\left(x(t_j) + x(t_{j+1})\right)}{2q} - \frac{q\left(f(t_j) + f(t_{j+1})\right)}{4}$$

In addition, the chosen interpolating polynomial for the state and control variables must satisfy the midpoint conditions for the differential equations as follows:

$$f(x_{app}(t_{c,j}), u_{app}(t_{c,j}), t_{c,j}) - \dot{x}_{app}(t_{c,j}) = 0 \qquad (18)$$

The optimal control problem now can be defined as a discretised problem as follows:

$$\min f(Y), \qquad (19)$$

subject to

$$\begin{aligned} & f(x_{app}(t), u_{app}(t), t) - \dot{x}_{app}(t_{c,j}) = 0 \\ & x_{app}(t_1) - x_1 = 0 \\ & \psi(x_{app}(t_k), t_k) = 0 \\ & C(x_{app}(t), u_{app}(t), t) \le 0 \\ & S(x_{app}(t), t) \le 0 \end{aligned} \qquad (20)$$

where x_{app}, u_{app} are the approximation of the state and control, constituting Y in (19). This above discretisation approach has been implemented in the DIRCOL [22,23] package which employed the sequential quadratic programming method SNOPT by Gill et al. [24].

4. Results and Discussion

In this example, the direct method based on DIRCOL package is used to solve the optimal trajectory. The missile assumes to be launched from ship and therefore the variation of terminal flight-path impact angle is used to investigate the different optimal exposure along the trajectory. The boundary conditions are given as follows:

$\gamma(0) = 0 \text{ deg}$ $\quad \gamma(t_f) = -45, -60, -70, -80, -90 \text{ deg}$

$V(0) = 272 \text{ m/s}$ $\quad V(t_f) = 310 \text{ m/s}$

$x(0) = 0 \text{ m}$ $\quad x(t_f) = 10000 \text{ m}$

$h(0) = 30 \text{m}$ $\quad h(t_f) = 0 \text{ m}$

Table 3. Performance index and final time

Case – final flight-path angle (deg)	Performance index (m sec)	Terminal time (sec)
-45	6210.11114	34.648947
-60	12654.6551	36.190966
-70	18160.0398	37.608110
-80	24511.1134	39.343370
-90	31596.3404	41.390514

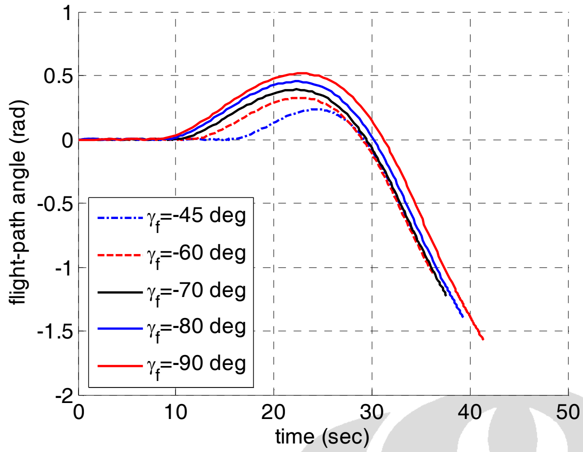

Figure 2. Flight-path angle versus time histories

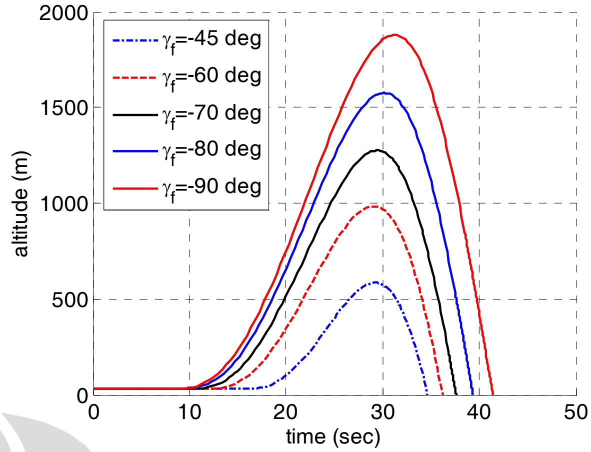

Figure 5. Altitude versus time histories

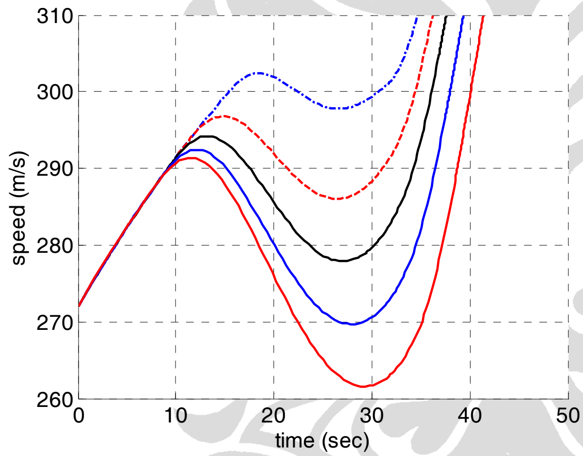

Figure 3. Speed versus time histories

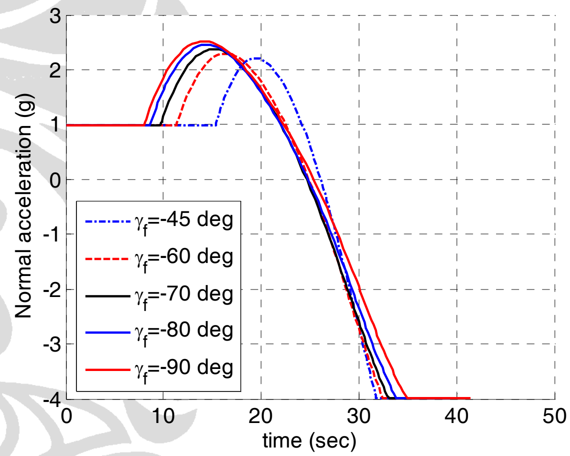

Figure 6. Normal acceleration versus time histories

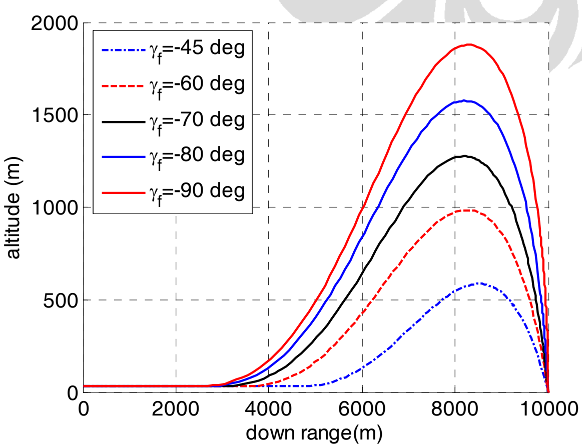

Figure 4. Altitude versus down-range histories

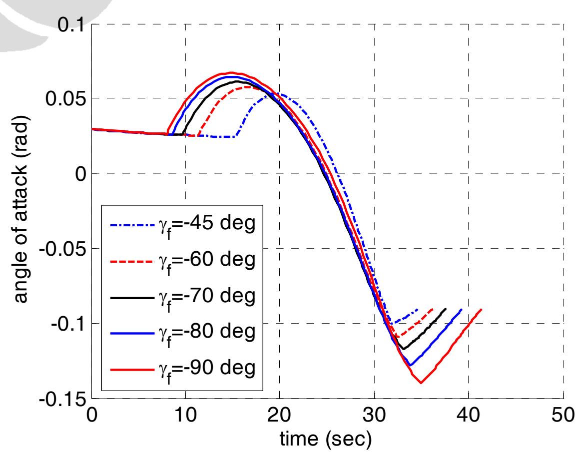

Figure 7. Angle of attack versus time histories

Based on Figures 2-7, an attempt is made to identify characteristic arcs of the trajectory, classify them according to the constraints active on them, and suggest physical/mathematical explanations for the observed behaviour. In this analysis the missile is assumed to be launched horizontally from the minimum altitude constraint.

The trajectory is split into three subintervals: level flight, climbing and diving. Each of the trajectory arcs corresponding to the subintervals is now discussed in turn.

First arc: minimum altitude flight. The missile launches at *h=30m* and therefore the altitude constraints are active directly at the start of the manoeuvre. In this case the altitude of the missile remains constant on the minimum value $h_{\min}$ (see Figures 4 and 5) until the missile must start climbing while the thrust is on the maximum value. The flight time of first arc depends to the final-speed γ_f.

Equation (1d) equals zero during this flight because the altitude remains constant. It means that the flight path angle γ equals zero because the velocity V is never equal to zero during flight (see Fig. 3). In addition, $\gamma(0) = 0$ causes the derivative of the flight path angle $\dot{\gamma}$ to be equal to zero (see Fig. 2). The dynamics equation (1) is therefore reduced as follows:

$$\dot{\gamma} = \frac{T-D}{mV}\sin\alpha + \frac{L}{mV}\cos\alpha - g = 0 \qquad (21a)$$

$$\dot{V} = \frac{T-D}{m}\cos\alpha - \frac{L}{m}\sin\alpha \qquad (21b)$$

$$\dot{x} = V \qquad (21c)$$

$$\dot{h} = 0 \qquad (21d)$$

We now consider the consequences of the right-hand side of equation (21a) being zero. This condition means that the normal acceleration *L/m* remains almost constant, because the angle of attack α is very small. The first term on the right-hand side of equation (21a) is small, because $\sin\alpha \approx \alpha \approx 0$ and we are left with $L/m \approx g$ due to $\cos\alpha \approx 1$. During this time speed increases, because for small α.

$$\dot{V} \approx \frac{T-D}{m} > 0, \quad \text{as } T > D$$

This in turn means that the angle of attack α slowly decreases in accordance with equation (4) and in order to maintain *L/m* approximately be equal to g (see Figures 6 and 7).

Second arc: climbing. The missile must climb eventually in order to achieve the final condition of the flight path angle V_f. Since altitude above $h_{\min}$ is penalised, the climb occurs as late as possible, so must be done sharply and last as short as possible. Hence, at the beginning of ascent the angle of attack must increase to facilitate a rapid nose up motion and the thrust has the maximum value.

During this time, the speed keeps decreasing while the altitude h increase. While rapid climbing is necessary, the missile should also turn over to begin its dive as soon as possible, so that the excess of altitude (above $h_{\min}$) is minimised.

Third arc: diving. At the end of the manoeuvre the missile should hit the target with a certain speed V_f. The speed during turnover is smaller than final speed V_f so the speed must increase and hence the thrust keeps on the maximum value. It means the thrust will facilitate the missile's arrival on the target as soon as possible.

At the beginning of diving the minimum normal acceleration constraints is active and keeps on saturation until the missile hits the target (see Fig. 6). Obviously, the altitude goes down to reach the target ($\gamma < 0 \rightarrow \dot{h} < 0$), while the speed goes up to satisfy the terminal speed condition V_f. Finally, the missile satisfies the terminal condition of the manoeuvre approximately t_f after firing. The performance index is smaller for the bigger final flight-path angle, similarly for the optimal manoeuvre time, see Table 3 for the detail.

4. Conclusion

The analysis of computational results for the minimum integrated altitude of a surface-to-surface missile guidance with varying terminal impact angle constraint is important from the operational viewpoint. Since the mission is to strike a fixed target while minimising the missile exposure to anti-air defences, one should consider both the type of target and the exposure of the missile during the manoeuvre. If the mission is to strike a bunker, it is important to hit the target with the maximum capability of the missile which means the missile must struck with vertical diving. If the target's prosecution may lead to collateral damage, then a more measured impact is advisable, so that the impact angle should be greater. It is always important to avoid anti-air defences during the manoeuvre, so optimal exposure must be taken into account.

The general trajectory is split into three subintervals. The first arc is level flight. The thrust is on the maximum value and the minimum altitude constraint is active. This arc is the most difficult one to compute because the pure state constraint is active. DIRCOL package can solve this arc and gives a good insight into the problem.

In the second arc the missile must climb in order to achieve the final condition. The thrust keeps on the maximum value while the speed decreases. The altitude increases to gain enough position for diving in the next arc.

The third arc is diving. The missile must gain the power to reach the target therefore the speed increase rapidly since the initial diving speed is lower than the final speed. The normal acceleration is saturated on the minimum value for this arc.

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